\( f(x_1, x_2) = 12x_1x_2(1-x_2) \quad 0 < x_1 < 1, \quad 0 < x_2 < 1, \quad \text{zero otherwise} \)

\( f_1(x_1) = \int_0^1 12x_1x_2(1-x_2) \, dx_2 = 2x_1 \quad \text{if} \quad 0 < x_1 < 1, \quad \text{zero otherwise} \)

\( f_2(x_2) = \int_0^1 12x_1x_2(1-x_2) \, dx_1 = 6x_2(1-x_2) \quad \text{if} \quad 0 < x_2 < 1, \quad \text{zero otherwise} \)

\[ f(x_1, x_2) = f_1(x_1) f_2(x_2) \quad \text{for all} \quad (x_1, x_2) \in S_{x_1} \times S_{x_2}. \quad X_1, X_2 \text{ are independent} \]

\[ f(x_1, x_2) = \frac{1}{16} \quad x_1 = 1, 2, 3, 4, \quad x_2 = 1, 2, 3, 4, \quad \text{zero elsewhere} \]

\[ f_1(x_1) = \frac{1}{2} \quad p(x_1, x_2) = \frac{1}{4} \quad \text{if} \quad x_1 = 1, 2, 3, 4, \quad \text{zero elsewhere} \]

\[ f_2(x_2) = \frac{1}{2} \quad p(x_1, x_2) = \frac{1}{4} \quad \text{if} \quad x_2 = 1, 2, 3, 4, \quad \text{zero elsewhere} \]

\[ f(x_1, x_2) = f_1(x_1) f_2(x_2) \quad \text{for all} \quad (x_1, x_2) \in S_{x_1} \times S_{x_2}. \quad X_1, X_2 \text{ are independent} \]

\[ f(x_1, x_2) = e^{-x_1-x_2} \quad 0 < x_1 < \infty, \quad 0 < x_2 < \infty \quad \text{zero elsewhere} \]

\[ f_1(x_1) = \int_0^\infty e^{-x_1-x_2} \, dx_2 = e^{-x_1} \quad \text{if} \quad 0 < x_1 < \infty, \quad \text{zero elsewhere} \]

\[ f_2(x_2) = \int_0^\infty e^{-x_1-x_2} \, dx_1 = e^{-x_2} \quad \text{if} \quad 0 < x_2 < \infty, \quad \text{zero elsewhere} \]

\[ f(x_1, x_2) = f_1(x_1) f_2(x_2) \quad \text{for all} \quad (x_1, x_2) \in S_{x_1} \times S_{x_2}. \quad X_1 \text{ and } X_2 \text{ are independent} \]

\[ M(t_1, t_2) = E[e^{t_1x_1 + t_2x_2}] = \int_0^\infty \int_0^\infty e^{t_1x_1 + t_2x_2} e^{-x_1-x_2} \, dx_1 \, dx_2 \]

\[ = \left( \int_0^\infty e^{-x_1(1-t_1)} \, dx_1 \right) \left( \int_0^\infty e^{-x_2(1-t_2)} \, dx_2 \right) \]

\[ = \left( \frac{1}{1-t_1} \right) \left( \frac{1}{1-t_2} \right) \quad \text{for} \quad t_1 < 1, \quad t_2 < 1. \]
2.5.6 Continue)
\[ E \left[ e^{t(x_1 + x_2)} \right] = M(t_1, t_2) \bigg|_{t_1 = t_2 = t} = \frac{1}{(1-t)^2}, \quad t < 1. \]

If \( Y = X_1 + X_2 \) then
\[ M_Y(t) = \frac{1}{(1-t)^2}. \]
\[ E(Y) = M'_Y(0) \bigg|_{t=0} = 2. \]
\[ E(Y^2) = M''_Y(0) \bigg|_{t=0} = 6, \quad V(Y) = 6 - 2^2 = 2. \]

2.5.9
\[ X: \text{Time of departure} \quad Y: \text{Time of travel} \]

\[ \begin{array}{c|c|c}
8:00 \text{AM} & 8:30 \text{AM} & 9:00 \text{AM} \\
\hline
0 & X & 30 \\
\hline
\end{array} \]

\[ x + y \text{ (Arrival Time)} \]

\[ f(x) = \frac{1}{30}, \quad 0 \leq x \leq 30, \quad 0 \text{ elsewhere}. \]

\[ f_2(y) = \frac{1}{10}, \quad 40 < y < 50, \quad 0 \text{ elsewhere}. \]

\[ x, y \text{ are independent}. \quad f(x,y) = \frac{1}{300}, \quad 0 \leq x \leq 30, \quad 40 < y < 50, \quad 0 \text{ elsewhere}. \]

\[ P[X + Y < 60] = P[Y < 60 - X] = \int_0^{60} \int_y^{60-y} \frac{1}{300} \, dx \, dy = \frac{1}{2}. \]

2.6.3
\( X_1, \ldots, X_n \) are independent r.v.s, each having pdf
\[ f_X(x) = 3(1-x)^2, \quad 0 < x < 1, \quad 0 \text{ elsewhere}. \]

and c.d.f.
\[ F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \int_0^x 3(1-t)^2 \, dt & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases} \]
Let $G(y)$ denote the c.o.d.f of $Y$.

\[
G(y) = P[Y \leq y] = 1 - P[Y > y] = 1 - P[X_i > y \quad i = 1, 2, 3] = 1 - \prod_{i=1}^{4} P[X_i > y] = 1 - \prod_{i=1}^{4} [1 - F_{X_i}(y)] = 1 - \prod_{i=1}^{4} [1 - F(x_i)](y)
\]

\[
= 1 - \left[1 - F_{X}(y)\right]^{3} = \begin{cases} 0 & \text{if } y < 0 \\ 1 - (1-y)^{12} & \text{if } 0 < y < 1 \\ 1 & \text{if } y \geq 1 \\ \end{cases}
\]

2.6.3 (Continued) $Y = \text{Maximum}(X_1, X_2, X_3)$.

$X_1$'s are independent rolls each having p.m.f

\[
f_{X}(x) = \frac{1}{6} \quad \text{if } x = 1, 2, \ldots, 6, \quad 0 \text{ elsewhere}
\]

and c.d.f.

\[
F_{X}(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{6} & \text{if } 1 \leq x < 2 \\ \frac{2}{6} & \text{if } 2 \leq x < 3 \\ \frac{3}{6} & \text{if } 3 \leq x < 4 \\ \frac{4}{6} & \text{if } 4 \leq x < 5 \\ \frac{5}{6} & \text{if } 5 \leq x \leq 6 \\ 1 & \text{if } x > 6
\end{cases}
\]

$Y = \text{Maximum}(X_1, X_2, X_3)$. Let $G(y)$ denote the c.o.d.f of $Y$.

\[
G(y) = P[Y \leq y] = P[X_i \leq y \quad i = 1, 2, 3] = \prod_{i=1}^{4} P(X_i \leq y)
\]

\[
= \prod_{i=1}^{4} F_{X_i}(y) = \left[F_{X}(y)\right]^3 = \begin{cases} 0 & \text{if } y < 1 \\ (\frac{1}{6})^3 & \text{if } 1 \leq y < 2 \\ (\frac{2}{6})^3 & \text{if } 2 \leq y < 3 \\ (\frac{3}{6})^3 & \text{if } 3 \leq y < 4 \\ (\frac{4}{6})^3 & \text{if } 4 \leq y < 5 \\ (\frac{5}{6})^3 & \text{if } 5 \leq y < 6 \\ 1 & \text{if } y \geq 6
\end{cases}
\]
then the p.m.f. of \( Y \) is

\[
\begin{align*}
\mathbb{P}(Y) = \begin{cases} 
\left( \frac{1}{6} \right)^3 & \text{if } y = 1 \\
\left( \frac{2}{6} \right)^3 - \left( \frac{1}{6} \right)^3 & \text{if } y = 2 \\
\left( \frac{3}{6} \right)^3 - \left( \frac{2}{6} \right)^3 & \text{if } y = 3 \\
\left( \frac{4}{6} \right)^3 - \left( \frac{3}{6} \right)^3 & \text{if } y = 4 \\
\left( \frac{5}{6} \right)^3 - \left( \frac{4}{6} \right)^3 & \text{if } y = 5 \\
\left( \frac{6}{6} \right)^3 - \left( \frac{5}{6} \right)^3 & \text{if } y = 6 \\
0 & \text{elsewhere.}
\end{cases}
\end{align*}
\]

That is

\[
\mathbb{P}(Y) = \left( \frac{y}{6} \right)^3 - \left( \frac{y-1}{6} \right)^3 \quad \text{if } y=1,2, \ldots , 6, \text{ zero elsewhere.}
\]

2.6.6 Suppose \( \mathbb{E}(X_i - \mu) \mid x_1, x_2, x_3 = b_2 (x_2 - \mu) + b_3 (x_3 - \mu) \).

Now, \( \mathbb{E}(X_i - \mu \mid x_1, x_2, x_3) = \int (x_i - \mu) f(x_i \mid x_1, x_2, x_3) \, dx_i \)

\[
= \frac{\int (x_i - \mu) f(x_i, x_1, x_2, x_3) \, dx_i}{f(x_1, x_2, x_3)}
\]

We have

\[
\int (x_i - \mu) f(x_i, x_1, x_2, x_3) \, dx_i = \left[ b_2 (x_2 - \mu) + b_3 (x_3 - \mu) \right] f(x_1, x_2, x_3)
\]

Multiply both sides with \((x_2 - \mu)\) and integrate w.r.t \(x_2\) and \(x_3\) we get

\[
\text{Cov}(X_i, X_2) = b_2 \sigma_2^2 + b_3 \text{Cov}(X_2, X_3) \quad (1)
\]

Multiply both sides of \((1)\) with \((x_3 - \mu)\) and integrate w.r.t \(x_2\) and \(x_3\), we get

\[
\text{Cov}(X_i, X_3) = b_2 \text{Cov}(X_2, X_3) + b_3 \sigma_3^2 \quad (2)
\]

Solve \((1)\) and \((2)\) simultaneously for \(b_2\) and \(b_3\).
\[ f(x_1, x_2, x_3) = e^{-x_1 - x_2 + x_3}, \quad 0 < x_1 < \infty, \quad c=1, \forall x, \text{ zero elsewhere} \]

\[ T : \quad Y_1 = \frac{x_1}{x_1 + x_2}, \quad Y_2 = \frac{x_1}{x_1 + x_2} - \frac{x_1 + x_2 + x_3}{x_1 + x_2 + x_3}, \quad Y_3 = x_1 + x_2 + x_3 \]

\[ T^{-1} : \quad X_1 = Y_1 Y_3, \quad X_2 = Y_2 Y_3 (1 - Y_1), \quad X_3 = Y_3 (1 - Y_2) \]

\[ J = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ -y_2 y_3 & y_1 (1 - Y_1) & y_2 (1 - Y_1) \\ 0 & -y_2 (1 - Y_2) & 0 \end{vmatrix} = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ 0 & y_3 & y_2 \\ 0 & 0 & 1 \end{vmatrix} = y_2 y_3 \]

\[ S_Y = \{(y_1, y_2, y_3) : 0 < y_1 < 1, 0 < y_2 < 1, 0 < y_3 < \infty \} \]

\[ g(y_1, x_2, x_3) = y_2 y_3 e^{-x_3}, \quad (y_1, y_2, y_3) \in S_Y, \text{ zero elsewhere} \]

\[ g_1(y_1) = 1, \quad 0 < y_1 < 1, \text{ zero elsewhere} \]

\[ g_2(y_2) = 2 y_2, \quad 0 < y_2 < 1, \text{ zero elsewhere} \]

\[ g_3(y_3) = \frac{1}{2} y_3^2 e^{-y_3}, \quad 0 < y_3 < \infty, \text{ zero elsewhere} \]

\[ g(y_1, y_2, y_3) = g_1(y_1) g_2(y_2) g_3(y_3) \]

\[ Y_1, Y_2, Y_3 \text{ are independent} \]

\[ f(x) = \begin{cases} \frac{1}{2}, & 0 < x < 1 \\ 0 & \text{ otherwise} \end{cases} \]

\[ T : \quad Y = X^2 \]

\[ T_1 : x = \sqrt{y}, \quad A_1 = \{ 1 < y \} \rightarrow B = \{ y \}, \quad x < 1, y = S_Y \]

\[ T_2 : x = -\sqrt{y}, \quad A_2 = \{ x < 1 \} \rightarrow B = \{ y \}, \quad x < 1, y = S_Y \]

\[ g(y) = \min \left\{ \left| \frac{d(f(y))}{dy} \right| + \left| \frac{d(f(-y))}{dy} \right|, \frac{1}{2\sqrt{y}} \right\} \]

\[ = \begin{cases} \frac{1}{2\sqrt{y}}, & 0 < y < 1 \\ 0, & \text{ otherwise} \end{cases} \]
2.7.3 \( f(x) = \frac{1}{4}, \quad -1 < x < 3, \quad \text{zero elsewhere.} \)

\[
T: \quad y = x^2
\]

\[
T_1': \quad x = \sqrt{y} \quad A_1: x: 0 < x < 1 \rightarrow B_1: y: 0 < y < 1
\]

\[
T_2': \quad x = -\sqrt{y} \quad A_2: x: -1 < x < 0 \rightarrow B_1: y: 0 < y < 1
\]

Thus for \( 0 < y < 1 \)

\[
g(y) = f(y) \left| \frac{d(\sqrt{y})}{dy} \right| + f(-\sqrt{y}) \left| \frac{d(-\sqrt{y})}{dy} \right| = \frac{1}{4\sqrt{y}}
\]

\[
T_3': \quad x = \sqrt{y} \quad A_3: \frac{x}{2}: 1 < x < 3 \rightarrow B_2: y: 1 < y < 9
\]

Thus for \( 1 < y < 9 \)

\[
g(y) = f(y) \left| \frac{d(\sqrt{y})}{dy} \right| = \frac{1}{8\sqrt{y}}
\]

\[
g(y) = \begin{cases} 
\frac{1}{4\sqrt{y}} & \text{if } 0 < y < 1 \\
\frac{1}{8\sqrt{y}} & \text{if } 1 < y < 9 \\
0 & \text{elsewhere.} 
\end{cases}
\]

2.7.4 \( f(x_1, x_2, x_3) = e^{-x_1 - x_2 - x_3} \quad \text{if } x_i < 0, \quad i = 1, 2, 3 \quad \text{zero elsewhere.} \)

\[
T_1: \quad y_1 = x_1 \\
T_2: \quad y_2 = x_1 + x_2 \\
T_3: \quad y_3 = x_1 + x_2 + x_3
\]

\[
S_y = \{y_1, y_2, y_3\} \quad 0 < y_1, y_2, y_3 < \infty
\]

\[
g(y_1, y_2, y_3) = e^{-y_1} \quad \text{if } 0 < y_1, y_2, y_3 < \infty \quad \text{zero elsewhere.}
\]
$X_1, X_2, X_3$ are i.i.d. with common m.g.f.

$$M(t) = \left[ \frac{3}{4} + \frac{1}{4} e^t \right]^2, \quad -\infty < t < \infty.$$  

a) For integers $n \geq 0$,  

$$ (a+b)^n = \sum_{x=0}^{\infty} (\binom{a}{x}) b^x a^{n-x} $$

Thus

$$M(t) = \left[ \frac{3}{4} + \frac{1}{4} e^t \right]^2 = \sum_{x=0}^{\infty} \binom{2}{x} \left( \frac{1}{4} e^t \right)^x \left( \frac{3}{4} \right)^{2-x} $$

$$= \sum_{x=0}^{\infty} e^{tx} \left( \frac{2}{x} \right) \left( \frac{1}{4} \right)^x \left( \frac{3}{4} \right)^{2-x} $$

Thus $X_i's$ have the common p.m.f.

$$p(x) = \binom{2}{x} \left( \frac{1}{4} \right)^x \left( \frac{3}{4} \right)^{2-x} \quad x = 0, 1, 2 \quad \text{zero elsewhere}$$

$$p \left( X_i = k \right) = \binom{2}{k} \left( \frac{1}{4} \right)^k \left( \frac{3}{4} \right)^{2-k}, \quad k = 0, 1, 2$$

b) $Y = X_1 + X_2 + X_3$.

$$M_Y(t) = E \left[ e^{tY} \right] = E \left[ e^{t(X_1 + X_2 + X_3)} \right] = \prod_{i=1}^{3} E \left[ e^{tX_i} \right] $$

$$= \left[ M(t) \right]^3 = \left[ \frac{3}{4} + \frac{1}{4} e^t \right]^6, \quad -\infty < t < \infty.$$  

$$= \sum_{y=0}^{\infty} \binom{6}{y} \left( \frac{1}{4} e^t \right)^y \left( \frac{3}{4} \right)^{6-y} $$

$$= \sum_{y=0}^{\infty} e^{ty} \left( \frac{6}{y} \right) \left( \frac{1}{4} \right)^y \left( \frac{3}{4} \right)^{6-y} $$

Thus the p.m.f. of $Y$ is

$$p_Y(y) = \begin{cases} \binom{6}{y} \left( \frac{1}{4} \right)^y \left( \frac{3}{4} \right)^{6-y} & y = 0, 1, \ldots, 6 \\ 0 & \text{elsewhere} \end{cases}$$