Confidence Intervals.

Definition:
Let $X_1, \ldots, X_n$ be a r.s from a distribution with parameter $\theta$. Let $U_1(X_1, \ldots, X_n)$ and $U_2(X_1, \ldots, X_n)$ be two functions of $X_1, \ldots, X_n$, not depending on any unknown parameters.

If the random interval $(U_1(X_1, \ldots, X_n), U_2(X_1, \ldots, X_n))$ covers the true parameter $\theta$ with probability $1 - \alpha$,
[i.e. if $P[U_1(X_1, \ldots, X_n) \leq \theta \leq U_2(X_1, \ldots, X_n)] = 1 - \alpha$]

then the computed interval $(U_1(X_1, \ldots, X_n), U_2(X_1, \ldots, X_n))$ for a given sample $X_1 = x_1, \ldots, X_n = x_n$ is called a confidence interval for $\theta$, with confidence coefficient $1 - \alpha$.

Definition: $Q(X_1, \ldots, X_n; \theta)$ is a pivotal quantity (or pivot) if the distribution of $Q(X_1, \ldots, X_n; \theta)$ is same for all values of $\theta$.

Confidence Intervals based on pivots:

1. $N(\mu, \sigma^2)$ $\sigma^2$ is known. Conf. interval for $\mu$,

Pivot $\bar{Z} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

$P[-z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}] = 1 - \alpha$

$\therefore P[\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}] = 1 - \alpha$

$\therefore U_1(X_1, \ldots, X_n) = \bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

$U_2(X_1, \ldots, X_n) = \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$
2. $N(\mu, \sigma^2)$ $\sigma^2$ is unknown.

CI for $\mu$:

$$Z = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim N(0,1)$$

$$W = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim \chi^2_{(n-1)}, \quad s^2 \sim \frac{(\bar{X} - \mu)^2}{\frac{(n-1)s^2}{n}}$$

$Z$ and $W$ are independent, since $\bar{X}$ and $S^2$ are independent.

$$T = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim t_{(n-1)}$$

$P_{\text{not}}$: $\frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim t_{(n-1)}$

$$P\left[-t_{\frac{\alpha}{2}, n-1} \leq \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \leq t_{\frac{\alpha}{2}, n-1}\right] = 1 - \alpha$$

$$P\left[-t_{\frac{\alpha}{2}, n-1} \leq \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \leq t_{\frac{\alpha}{2}, n-1}\right] = 1 - \alpha$$

$$P\left[\frac{-t_{\frac{\alpha}{2}, n-1}}{\frac{s}{\sqrt{n}}} \leq \mu \leq \frac{t_{\frac{\alpha}{2}, n-1}}{\frac{s}{\sqrt{n}}}\right] = 1 - \alpha$$

$$U_1 (X_1, \ldots, X_n) = \bar{X} - t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}}$$

$$U_2 (X_1, \ldots, X_n) = \bar{X} + t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}}$$

3. $N(\mu, \sigma^2)$ $\mu$ is unknown. CI for $\sigma^2$.

$$P_{\text{not}}: \bar{X}^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

$$P\left[\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}, n-1}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}, n-1}}\right] = 1 - \alpha$$
4. \( \Gamma(\alpha, \beta) \). \( \alpha \) is a known integer. CI for \( \beta \):

\[ Y = \sum X_i \sim \Gamma(\alpha, \beta) \]

\[ m_Y(1) = \frac{1}{(1-\beta)^{\alpha}} \]

\[ W = \frac{2Y}{\beta} \]

\[ m_W(1) = m_Y \left( \frac{2 \beta}{\beta} \right) = \frac{1}{(1-2\beta)^{\alpha}} \]

\[ W \sim \chi^2_{2\alpha} \]

Pivot:

\[ \chi^2 = \frac{2 \sum X_i}{\beta} \sim \chi^2_{2\alpha} \]

\[ P \left[ \chi^2_{1-\alpha, 2\alpha} \leq \frac{2 \sum X_i}{\beta} \leq \chi^2_{\alpha, 2\alpha} \right] = 1-\alpha \]

\[ P \left[ \frac{2 \sum X_i}{\chi^2_{1-\alpha, 2\alpha}} \leq \beta \leq \frac{2 \sum X_i}{\chi^2_{\alpha, 2\alpha}} \right] = 1-\alpha \]

5. \( Y \sim \text{Binomial}(n, p) \)

Large sample CI for \( p \):

Let \( \hat{p} = \frac{Y}{n} \)

a) Pivot:

\[ z = \frac{Y-np}{\sqrt{np(1-p)}} \xrightarrow{D} N(0, 1) \]

\[ P \left[ -z_{\alpha} \leq \frac{\hat{p} - p}{\sqrt{p(1-p)}} \leq z_{\alpha} \right] = 1-\alpha \]

\[ P \left[ \left( \frac{\hat{p} - p}{\sqrt{p(1-p)}} \right)^2 \leq \frac{2}{1-p} \right] = 1-\alpha \]

\[ P \left[ (1 + \frac{\hat{p}^2}{n}) \leq (2 \hat{p} + \frac{2\hat{p}^2}{n}) \right] = 1-\alpha \]

The coefficient of \( \hat{p}^2 \) in the quadratic is positive, hence the quadratic opens upward and thus the inequality
is satisfied if \( p \) lies between the two roots of the quadratic. These two roots are:

\[
\left( 2 \hat{p} + \frac{e^2}{n} \right) \pm \sqrt{\left( 2 \hat{p} + \frac{e^2}{n} \right)^2 - 4 \hat{p} \left( 1 + \frac{e^2}{n} \right)} = \frac{2 \hat{p} - \frac{e^2}{n}}{1 + \frac{e^2}{n}}
\]

and the roots define the endpoints of the CI for \( p \).

b) **Pivot:** \( Z = \frac{\hat{p} - p}{\sqrt{\hat{p}(1-\hat{p})}} \sim N(0,1) \)

Using the results in Chapter 4.

\[
P \left( \hat{p} - Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right) \leq p \leq \hat{p} + Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \Rightarrow 1 - \alpha
\]

6. **Independent Normal distributions:** \( N(\mu, \sigma_1^2), N(\mu, \sigma_2^2) \)

\( \sigma_1, \sigma_2 \) are known. **CI for \( \theta = \mu_1 - \mu_2 \):**

**Pivot:** \( Z = \frac{(\bar{X}_1 - \bar{X}_2) - \theta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1) \)

**CI for \( \mu_1 - \mu_2 \):** \( (\bar{X}_1 - \bar{X}_2) \pm Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \)

where \( \bar{X}_1 \) is the sample mean of the first of size \( n_1 \) from \( N(\mu_1, \sigma_1^2) \) and \( \bar{X}_2 \) is the sample mean of the second of size \( n_2 \) from \( N(\mu_2, \sigma_2^2) \).
7. **Independent Normal Distributions:** \( N(\mu_1, \sigma_1^2), N(\mu_2, \sigma_2^2) \)

\[ \sigma_1 = \sigma_2 = \sigma, \quad \sigma^2 \text{ is unknown.} \]

CI for \( \theta = \mu_1 - \mu_2 \):

\[ Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}}} \sim N(0,1) \]

\[ \frac{(n_1-1)s_1^2}{\sigma^2} \sim \chi^2(n_1-1), \quad \frac{(n_2-1)s_2^2}{\sigma^2} \sim \chi^2(n_2-1) \]

\[ W = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{\sigma^2} \sim \chi^2(n_1+n_2-2) \]

\[ T = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}} \]

\[ \text{is the pivot.} \]

CI for \( \mu_1 - \mu_2 \) is then

\[ (\bar{x}_1 - \bar{x}_2) \pm t_{\frac{\alpha}{2}, n_1+n_2-2} \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} \left( \frac{1}{n_1} + \frac{1}{n_2} \right)} \]
8) $Y_1 \sim \text{Binomial} \left(n_1, p_1 \right)$
$Y_2 \sim \text{Binomial} \left(n_2, p_2 \right)$ \hspace{1cm} $Y_1, Y_2$ are independent.

Large Sample Conf. Int. for $\Theta = p_1 - p_2$.

Let $\hat{p}_1 = \frac{Y_1}{n_1}$ \hspace{1cm} $\hat{p}_2 = \frac{Y_2}{n_2}$

Pivot: $Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}} \longrightarrow N(0,1)$

Conf. Int.: $(\hat{p}_1 - \hat{p}_2) \pm Z_{\alpha/2} \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$

**Theorem**: Let $Y$ be a discrete statistic with

$d.f. G(y | \theta)$. Let $0 \leq \alpha \leq 1$ be a fixed value. Suppose $u_1(y)$ and $u_2(y)$ can be defined as follows:

a) If $G(y | \theta)$ is a decreasing function of $\theta$ for each $y$, define $u_1(y)$ and $u_2(y)$ by

$$P[ Y \leq y | \theta = u_1(y) ] = \frac{\alpha}{2}, \hspace{1cm} P[ Y \geq y | \theta = u_2(y) ] = \frac{\alpha}{2}$$

b) If $G(y | \theta)$ is an increasing function of $\theta$ for each $y$, define $u_1(y)$ and $u_2(y)$ by

$$P[ Y \geq y | \theta = u_2(y) ] = \frac{\alpha}{2}, \hspace{1cm} P[ Y \leq y | \theta = u_1(y) ] = \frac{\alpha}{2}$$

Then $U_1(y), U_2(y)$ are $\text{Random Variables}$ such that

$$P \left[ U_1(y) \leq \Theta \leq U_2(y) \right] = 1 - \alpha,$$

and thus $[U_1(y), U_2(y)]$ is a confidence interval for $\Theta$. 

Example: Suppose $Y_n$ Poisson $(\theta)$

$$G(y|\theta) = \sum_{k=0}^{y} \frac{e^{-\theta} \theta^k}{k!}$$

is a decreasing function of $\theta$ for each $y$. Thus $[u_1, u_2]$ is a $1-\alpha$ confidence interval for $\theta$, where:

$$\frac{\sum_{k=0}^{y} e^{-u_1} u_1^k}{k!} = k \quad \text{and} \quad \frac{\sum_{k=y}^{\infty} e^{-u_1} u_1^k}{k!} = \frac{\alpha}{2}$$

$u_1$ and $u_2$ can be found by using the following result:

For $c > 1$, $\ldots$

$$\Pr[y \leq c-1] = \sum_{k=0}^{c-1} \frac{e^{-\theta} \theta^k}{k!} = \int_0^{\infty} \frac{1}{p(c)} \frac{u^{c-1} e^{-u}}{u} du$$

Let $w = 2u$

$$\int_0^{\infty} \frac{1}{p(2c)} \frac{w^{2c-1} e^{-w/2}}{w} dw = p \left[ \chi^2_{2c} \geq 2\theta \right]$$

if $Y_n$ Poisson $(\theta)$ then $\Pr[y \leq c-1] = p \left[ \chi^2_{2c} \geq 2\theta \right]$

By using the above link between the Poisson and Chi-Square distribution we have:
\[
\frac{\alpha}{2} = P[Y \leq y_1 | \theta = u_1] = P[X_{2(y+1)}^2 \geq 2u_1]
\]

Thus
\[
2u_1 = X_{2(y+1)}, \frac{\alpha}{2}
\]

\[
u_1 = \frac{2u_1}{\alpha} = P[Y \leq y_1 | \theta = u_1] = P[X_{2y}^2 \geq 2u_1]
\]

Thus
\[
u_1 = \frac{X_{2y+1}^2}{\alpha}
\]

Confidence interval for \( \theta \):
\[
\left[ \frac{X_{2y}, 1 - \frac{\alpha}{2}}{\alpha}, \frac{X_{2(y+1)}, \frac{\alpha}{2}}{\alpha} \right]
\]

Note that we define:
\[
X_{0.1-\frac{\alpha}{2}}^2 = 0
\]

Example: Suppose \( X_1, \ldots, X_n \) is a r.v. from Poisson (\( \theta \))

\[
Y = \sum_{i=1}^{n} X_i \sim \text{Poisson} (n \theta)
\]

Using the result in the previous example, confidence

interval for \( n \theta \) is

\[
\left[ \frac{X_{2y}, 1 - \frac{\alpha}{2}}{\alpha}, \frac{X_{2(y+1)}, \frac{\alpha}{2}}{\alpha} \right]
\]

thus the confidence interval for \( \theta \) is

\[
\left[ \frac{X_{2y}, 1 - \frac{\alpha}{2}}{2n}, \frac{X_{2(y+1)}, \frac{\alpha}{2}}{2n} \right]
\]
Example: Suppose $Y \sim \text{Binomial} \ (n, \theta)$.

\[
G(y | \theta) = \sum_{k=0}^{y} \binom{n}{k} \theta^k (1-\theta)^{n-k}
\]

is a decreasing function of $\theta$ for each $y$. (u_1, u_2)$ is a confidence interval for $\theta$, where

\[
P(Y \leq y | \theta = u_2) = \frac{\alpha}{2} \quad \text{and} \quad P(Y \geq y | \theta = u_1) = \frac{\alpha}{2}
\]

First note the following results:

1) if $Y \sim \text{Binomial} \ (n, \theta)$ then

\[
P(Y \leq c-1) = P(W > 0)
\]

where $W \sim \Beta(n-c, \alpha_2 = n-c+1)$

2) if $W \sim \Beta(\alpha_1, \alpha_2)$ then

\[
\frac{\alpha_1}{\alpha_2} \left( \frac{w-1}{n-w} \right) \sim F_{2(n-c+1), 2c}
\]

Combining (1) and (2) we have the following relationship between the Binomial and F-distributions.

If $Y \sim \text{Binomial} \ (n, \theta)$ then

\[
P(Y \leq c-1) = P \left( F_{2(n-c+1), 2c} \leq \left( \frac{c}{n-c+1} \right) \left( \frac{1}{\theta - 1} \right) \right)
\]

Using the above link,

\[
\frac{\alpha}{2} = P(Y \leq y | \theta = u_2) = P \left( F_{2(n-y), 2(y+1)} \leq \left( \frac{y+1}{n-y} \right) \left( \frac{1}{u_2 - 1} \right) \right)
\]
Thus \[
\left( \frac{y+1}{n-y} \right) \left( \frac{1}{q_2 - 1} \right) = F_{2(n-y), 2(y+1), 1 - \frac{\alpha}{2}}^0
\]

\[
U_2 = \frac{1}{\left( \frac{n-y}{y+1} \right) F_{2(n-y), 2(y+1), 1 - \frac{\alpha}{2}}^0 + 1}
\]

Using \[
F_{k_1, k_2, 1 - \frac{\alpha}{2}} = \frac{1}{F_{k_2, k_1, \frac{\alpha}{2}}}
\]

\[
F_{2(n-y), 2(y+1), 1 - \frac{\alpha}{2}} = \frac{1}{F_{2(y+1), 2(n-y), \frac{\alpha}{2}}}
\]

Then \[
U_2 = \frac{1}{\left( \frac{n-y}{y+1} \right) \frac{1}{F_{2(y+1), 2(n-y), \frac{\alpha}{2}}} + 1}
\]

\[
U_2 = \frac{\left( \frac{y+1}{n-y} \right) F_{2(y+1), 2(n-y), \frac{\alpha}{2}}}{1 + \left( \frac{y+1}{n-y} \right) F_{2(y+1), 2(n-y), \frac{\alpha}{2}}}
\]

\[
U_2 = \frac{1}{\frac{K_0 + 1}{K_0}} = \frac{f}{K + f} = \frac{f}{1 + \frac{1}{K} f}
\]

= \frac{\left( \frac{y+1}{n-y} \right) f}{1 + \left( \frac{y+1}{n-y} \right) f}
\[ \frac{\alpha \cdot \theta}{2} = \Phi \left( \frac{Y_{n-1}}{\theta = u_1} \right) \]

\[ 1 - \frac{\alpha \cdot \theta}{2} = \Phi \left( Y_{n-1} \right) \]

\[ = \Phi \left( \frac{F_{2(n-y+1), 2Y}}{u_1, 1} \right) \]

\[ = \frac{\alpha \cdot \theta}{2} \]

\[ \frac{Y_{n-y+1}}{u_1, 1} \left( \frac{1}{u_1, 1} \right) = F_{2(n-y+1), 2Y, \frac{\alpha \cdot \theta}{2}} \]

\[ U_1 = \frac{1}{1 + \left( \frac{n-y+1}{Y} \right) F_{2(n-y+1), 2Y, \frac{\alpha \cdot \theta}{2}}} \]

Confidence interval for \( \theta \):

\[ \left[ \frac{1}{1 + \left( \frac{n-y+1}{Y} \right) F_{2(n-y+1), 2Y, \frac{\alpha \cdot \theta}{2}}} \right] \]

\[ \frac{\left( \frac{Y+1}{n-2} \right) F_{2(Y+1), 2(n-y), \frac{\alpha \cdot \theta}{2}}} {1 + \left( \frac{2+1}{n-2} \right) F_{2(Y+1), 2(n-y), \frac{\alpha \cdot \theta}{2}}} \]
Example: Let $X_1, \ldots, X_n$ be a r.v.s from a distribution with p.d.f.
$$f(x|\theta) = \begin{cases} e^{-(x-\theta)} & \text{if } x \geq \theta \\ 0 & \text{elsewhere} \end{cases}$$

Let $Y = \min (X_1, \ldots, X_n)$ then $Y$ is a sufficient statistic for $\theta$.
$$g(y|\theta) = \begin{cases} y & \text{if } y \geq \theta \\ 0 & \text{elsewhere} \end{cases}$$

$$G(y|\theta) = \int_0^y t e^{-n(t-\theta)} \, dt = 1 - e^{-n(y-\theta)}$$

is a decreasing function of $\theta$. 

$[u_1, u_2]$ is a confidence interval for $\theta$, where

$$\frac{\alpha}{2} = P[\frac{Y \leq y}{\theta = u_2}] = \int_{u_2}^{y} n e^{-n(t-u_2)} \, dt = 1 - e^{-n(y-u_2)}$$

and

$$\frac{\alpha}{2} = P[\frac{Y \geq y}{\theta = u_1}] = \int_{y}^{\infty} n e^{-n(t-u_1)} \, dt = e^{-n(y-u_1)}$$

Thus

$$u_1 = y + \frac{1}{n} \log \frac{\alpha}{2}$$

$$u_2 = y + \frac{1}{n} \log (1 - \frac{\alpha}{2})$$

Confidence interval for $\theta$:

$$\left[ y + \frac{1}{n} \log \frac{\alpha}{2}, \ y + \frac{1}{n} \log (1 - \frac{\alpha}{2}) \right]$$
Confidence interval for \( \theta \), based on a pivot.

\[ Y = \min(X_1, \ldots, X_n) \quad g(y; \theta) = n \cdot e^{-n(y - \theta)} \quad \theta > 0 \]

Pivot: \( W = Y - \theta \) \( h(w; \theta) = n \cdot e^{-nw} \quad \text{if} \quad w > 0 \).

Let \( W_{1/2} \) and \( W_{1-\alpha} \) be such that

\[ P \left[ W > W_{1/2} \right] = \frac{\alpha}{2} \quad P \left[ W > W_{1-\alpha} \right] = 1 - \frac{\alpha}{2} \text{ as usual.} \]

Then

\[ P \left[ W_{1-\alpha/2} \leq W \leq W_{1/2} \right] = 1 - \alpha \]

\[ P \left[ W_{1-\alpha/2} \leq Y - \theta \leq W_{1/2} \right] = 1 - \alpha \]

\[ P \left[ Y - W_{1/2} \leq \theta \leq Y - W_{1-\alpha/2} \right] = 1 - \alpha. \]

\[ [Y - \frac{W_{1/2}}{n}, Y - \frac{W_{1-\alpha/2}}{n}] \] is a confidence interval for \( \theta \).

where

\[ \frac{\alpha}{2} \int_{-\frac{W_{1/2}}{n}}^{\infty} n \cdot e^{-nw} \, dw = - e^{-\frac{\alpha}{2} W_{1/2}} = e^{-n \cdot \frac{\alpha}{2}} \]

\[ n \cdot \frac{\alpha}{2} = \log \frac{\alpha}{2} \]

\[ \frac{W_{1/2}}{n} = - \left( \frac{1}{n} \right) \log \frac{\alpha}{2} \]

Similarly

\[ W_{1-\alpha/2} = - \left( \frac{1}{n} \right) \log (1 - \frac{\alpha}{2}) \]

Thus

\[ [Y + \frac{1}{n} \log \frac{\alpha}{2}, Y + \frac{1}{n} \log (1 - \frac{\alpha}{2})] \] is a confidence interval for \( \theta \), same as before. [They do not necessarily the same based on the theorem.]