Solutions:

Problem #1: \( X \sim f(x) \) with

\[ f(x) = 3x^2, \quad \text{for } 0 < x < 1. \]

CDF of \( X \):

\[ F(x) = \int_{0}^{x} 3t^2 dt = x^3. \]

The first quartile:

\[ x_{.25} : \quad F(x_{.25}) = .25 \]

\[ x_{.25}^3 = .25 \]

\[ x_{.25} = \left( .25 \right)^{\frac{1}{3}} = .63 \]

The median \( x_{.5} \):

\[ F(x_{.5}) = .5 \]

\[ x_{.5}^3 = .5 \]

\[ x_{.5} = \left( .5 \right)^{\frac{1}{3}} = .79 \]

The third quartile:

\[ x_{.75} : \quad F(x_{.75}) = .75 \]

\[ x_{.75}^3 = .75 \]

\[ x_{.75} = \left( .75 \right)^{\frac{1}{3}} = .91 \]
The mean of $X$: \[ \mu = E(X) = \int_{-2}^{1} x f(x) \, dx \]
\[ \mu = \int_{0}^{1} x \cdot 3x^2 \, dx = 3 \int_{0}^{1} x^3 \, dx = \frac{3}{4} x^4 \bigg|_{x=0}^{x=1} = \frac{3}{4} \]

The variance of $X$: \[ \sigma^2 = E(X - \mu)^2 = E(X^2) - \mu^2 \]
\[ \sigma^2 = E(X^2) - \left( \frac{3}{4} \right)^2 = \int_{0}^{1} x^2 \cdot 3x^2 \, dx - \left( \frac{3}{4} \right)^2 = \]
\[ = 3 \int_{0}^{1} x^4 \, dx - \frac{9}{16} = \frac{3}{5} \cdot \frac{9}{16} = 0.375 \]

Problem #2: $T_i$, $i=1,2,3,4$ are independent r.v.'s with means $\mu_i$: $\mu_1=3$, $\mu_2=10$, $\mu_3=8$, $\mu_4=5$ and variances: 2, 3, 4 and 6 respectively.

Let $Y = T_1 + T_2 + T_3 + T_4$, then
\[ E(Y) = 3 + 10 + 8 + 5 = 26 \]
\[ \text{var}(Y) = 2 + 3 + 4 + 6 = 15 \]
Problem #3: Let us calculate the sample mean and the sample variance:

\[ \bar{x} = 9.52 \quad ; \quad \sigma^2 = 1.142 \]

Sample s.d.:
\[ s = \sqrt{1.142} = 1.0686 \]

\[ n = 5 < 30, \quad 1 - \alpha = .95, \quad \alpha_2 = .025 \]

From table C.6 with d.f. = 4
\[ t_{0.025, 4} = 2.776 \]

95% CI for \( \mu \):
\[ \bar{x} \pm 2.776 \cdot \frac{s}{\sqrt{5}} \]

or
\[ 9.52 \pm 2.776 \cdot \frac{1.0686}{\sqrt{5}} \]

or
\[ 8.19 \leq \mu \leq 10.85 \]

Problem #4: \( n = 100 > 30, \quad \hat{p} = \frac{8}{100} = .08 \)

98% CI for proportion \( p \):
\[ \hat{p} \pm z_{0.01} \sqrt{\frac{\hat{p}(1-\hat{p})}{100}} \]

\[ 1 - \alpha = .98, \quad \alpha_2 = .01, \quad z_{0.01} = 2.326 \]

CI \( p \):
\[ .08 \pm 2.326 \times \sqrt{\frac{.08(1-.08)}{100}} \]

or
\[ .08 \pm .0631 \]

or
\[ .017 \leq p \leq .143 \]
Problem #5:  \( n_1 = 14 < 30 \quad \bar{X} = 17 \quad S_1^2 = 1.5 \)
\( n_2 = 14 < 30 \quad \bar{Y} = 19 \quad S_2^2 = 1.8 \)

Let us use the table C.6 with
\[ d.f. = n_1 + n_2 - 2 = 14 + 14 - 2 = 26 \]

\( 1 - \alpha = .99 \quad \alpha/2 = .005 \quad t_{.005, 26} = 2.779 \)

\( 99\% \) CI for \( \mu_1 - \mu_2 \):

\[ \bar{X} - \bar{Y} \pm 2.779 \sqrt{S_p^2 \left( \frac{1}{n_1} + \frac{1}{n_2} \right)} \]

Where \( S_p^2 = \frac{(14-1)(1.5) + (14-1)(1.8)}{14 + 14 - 2} = 1.65 \)

i.e. \( 99\% \) CI for \( \mu_1 - \mu_2 \):

\[ 17 - 19 \pm 2.779 \sqrt{1.65 \left( \frac{1}{14} + \frac{1}{14} \right)} \]

\[ -2 \pm 1.3492 \]

or

\[ -3.35 \leq \mu_1 - \mu_2 \leq -0.65 \]
Problem #6: \( X = \text{Normal}(\mu = 31, \sigma^2 = (.2)^2) \)
The standardized Normal (0, 1) is
\[
Z = \frac{X - 31}{.2}
\]
a) \( P(X > 30.5) = P\left(Z > \frac{30.5 - 31}{.2}\right) = \)
\[= P(Z > -2.5) = 1 - P(Z \leq -2.5) = \]
\[= 1 - \Phi(-2.5) = 1 - \left(1 - \Phi(2.5)\right) = \Phi(2.5) = \boxed{.9938} \]
b) \( P(30.5 < X < 31.5) = P\left(\frac{30.5 - 31}{.2} < Z < \frac{31.5 - 31}{.2}\right) \)
\[= P(-2.5 < Z < 2.5) = \Phi(2.5) - \Phi(-2.5) = \]
\[= \Phi(2.5) - (1 - \Phi(2.5)) = 2\Phi(2.5) - 1 =
2 \times (.9938) - 1 = \boxed{.9876} \]
c) \( p = P(X < 30.4) \) - probability that the tire is under inflated
\[p = P\left(Z < \frac{30.4 - 31}{.2}\right) = P(Z < -3) = \Phi(-3) = \]
\[= 1 - \Phi(3) = 1 - .9987 = \boxed{.0013} \]
Let \( Y = \# \) of under inflated tires out of 4
\[Y = \text{Bin}(n=4, p=0.0013) \]
\[ P(Y \geq 1) = 1 - P(Y = 0) = \]
\[ = 1 - \binom{4}{0} (0.0013)^0 (1 - 0.0013)^4 = 0.0052 \]

**Problem #7**: \( n = 20 \)

For \( i = 1, 2, \ldots, 20 \): \( X_i \) are independent \( \text{Normal}(4, \sigma^2) \)

\[(1-\alpha) \times 100\% \ CI \ for \ \sigma^2: \]
\[ \frac{(n-1) S^2}{\chi^2_{\alpha/2, n-1}} \leq \sigma^2 \leq \frac{(n-1) S^2}{\chi^2_{1-\alpha/2, n-1}} \]

\[ 1 - \alpha = .98, \ \alpha = .02, \ \frac{\alpha}{2} = .01 \]

From the Table C.5 with \( df = 20 - 1 = 19 \)

\[ \chi^2_{.01, 19} = 36.191 \quad \text{and} \quad \chi^2_{.99, 19} = 7.633 \]

\[ 98\% \ CI \ for \ \sigma^2: \]
\[ \frac{19 \times S^2}{\chi^2_{.01, 19}} \leq \sigma^2 \leq \frac{19 \times S^2}{\chi^2_{.99, 19}} \]

i.e. with \( S^2 = 16 \), \( CI_{\sigma^2} \) is:

\[ \frac{19 \times 16}{36.191} \leq \sigma^2 \leq \frac{19 \times 16}{7.633} \]

or

\[ 8.4 \leq \sigma^2 \leq 39.8 \]
Problem #8: \( n = 49 > 0 \)

According to the CLT, \( \bar{X} \approx N(\mu = 4, \sigma^2 = \frac{4^2}{49}) \)

Since each \( X_i \) has \( \exp(\beta = 4) \):

\[
\begin{align*}
\mathbb{E}(X_i) &= \beta = 4 \\
\text{Var}(X_i) &= \beta^2 = 4^2
\end{align*}
\]

The standardized r.v.

\[
Z = \frac{\bar{X} - 4}{\sqrt{\frac{4^2}{49}}} = \frac{\bar{X} - 4}{\frac{4}{7}}
\]

and

\[
P(3.1 < \bar{X} < 4.6) = P\left(\frac{3.1 - 4}{\frac{4}{7}} < Z < \frac{4.6 - 4}{\frac{4}{7}}\right)
\]

\[
= P(-1.575 < Z < 1.05)
\]

\[
= \Phi(1.05) - \Phi(-1.575)
\]

\[
= \Phi(1.05) - (1 - \Phi(1.575))
\]

\[
= .8531 - (1 - .9429) = .796
\]