Black-Scholes option pricing formula: Part I

Homework: Mikosch, T. (1998). Elementary Stochastic Calculus: Ch. 1, Sec.3; Ch. 4, Sec. 1
1.3.2 Processes Derived from Brownian Motion

The purpose of this section is to get some feeling for the distributional and pathwise properties of Brownian motion. If you want to start with Chapter 2 on stochastic calculus as soon as possible, you can easily skip this section and return to it whenever you need a reference to a property or definition.

Various Gaussian and non-Gaussian stochastic processes of practical relevance can be derived from Brownian motion. Below we introduce some of those processes which will find further applications in the course of this book. As before, $B = (B_t, t \in [0, \infty))$ denotes Brownian motion.

Example 1.3.5 (Brownian bridge)

Consider the process

$$X_t = B_t - t B_1, \quad 0 \leq t \leq 1.$$  

Obviously,

$$X_0 = B_0 - 0 B_1 = 0 \quad \text{and} \quad X_1 = B_1 - 1 B_1 = 0.$$  

For this simple reason, the process $X$ bears the name (standard) Brownian bridge or tied down Brownian motion. A glance at the sample paths of this “bridge” (see Figure 1.3.4) may or may not convince you that this name is justified.

Figure 1.3.4 A sample path of the Brownian bridge.
1.3. BROWNIAN MOTION

Using the formula for linear transformations of Gaussian random vectors (see p. 18), one can show that the fidis of $X$ are Gaussian. Verify this! Hence $X$ is a Gaussian process. You can easily calculate the expectation and covariance functions of the Brownian bridge:

$$\mu_X(t) = 0 \quad \text{and} \quad c_X(t,s) = \min(t,s) - t s, \quad s,t \in [0,1].$$

Since $X$ is Gaussian, the Brownian bridge is characterized by these two functions.

The Brownian bridge appears as the limit process of the normalized empirical distribution function of a sample of iid uniform $U(0,1)$ random variables. This is a fundamental result from non-parametric statistics; it is the basis for numerous goodness-of-fit tests in statistics. See for example Shorack and Wellner (1986).

![Figure 1.3.6](image.png)

**Figure 1.3.6** A sample path of Brownian motion with drift $X_t = 20 B_t + 10 t$ on $[0,100]$. The dashed line stands for the drift function $\mu_X(t) = 10 t$.

**Example 1.3.7** (Brownian motion with drift)

Consider the process

$$X_t = \mu t + \sigma B_t, \quad t \geq 0,$$

for constants $\sigma > 0$ and $\mu \in \mathbb{R}$. Clearly, it is a Gaussian process (why?) with expectation and covariance functions

$$\mu_X(t) = \mu t \quad \text{and} \quad c_X(t,s) = \sigma^2 \min(t,s), \quad s,t \geq 0.$$
The expectation function $\mu_X(t) = \mu t$ (the deterministic "drift" of the process) essentially determines the characteristic shape of the sample paths; see Figure 1.3.6 for an illustration. Therefore $X$ is called Brownian motion with (linear) drift.

With the fundamental discovery of Bachelier in 1900 that prices of risky assets (stock indices, exchange rates, share prices, etc.) can be well described by Brownian motion, a new area of applications of stochastic processes was born. However, Brownian motion, as a Gaussian process, may assume negative values, which is not a very desirable property of a price. In their celebrated papers from 1973, Black, Scholes and Merton suggested another stochastic process as a model for speculative prices. In Section 4.1 we consider their approach to the pricing of European call options in more detail. It is one of the promising and motivating examples for the use of stochastic calculus.

**Example 1.3.8 (Geometric Brownian motion)**

The process suggested by Black, Scholes and Merton is given by

$$X_t = e^{\mu t + \sigma B_t}, \quad t \geq 0,$$

i.e. it is the exponential of Brownian motion with drift; see Example 1.3.7. Clearly, $X$ is not a Gaussian process (why?).

For the purpose of later use, we calculate the expectation and covariance functions of geometric Brownian motion. For readers, familiar with probability theory, you may recall that for an $N(0, 1)$ random variable $Z,

$$Ee^{\lambda Z} = e^{\lambda^2/2}, \quad \lambda \in \mathbb{R}.$$  \hspace{1cm} (1.15)

It is easily derived as shown below:

$$Ee^{\lambda Z} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{\lambda z} e^{-z^2/2} \, dz$$

$$= e^{\lambda^2/2} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-(z - \lambda)^2/2} \, dz$$

$$= e^{\lambda^2/2}.$$

Here we used the fact that $(2\pi)^{-1/2} \exp\{- (z - \lambda)^2/2\}$ is the density of an $N(\lambda, 1)$ random variable.

From (1.15) and the self-similarity of Brownian motion it follows immediately that

$$\mu_X(t) = e^{\mu t} Ee^{\sigma B_t} = e^{\mu t} Ee^{\sigma t^{1/2} B_1} = e^{(\mu + 0.5 \sigma^2) t}.$$  \hspace{1cm} (1.16)
Figure 1.3.9 Sample paths of geometric Brownian motion $X_t = \exp\{0.01t + 0.01B_t\}$ on $[0,10]$, the expectation function $\mu_X(t)$ (dashed line) and the graphs of the functions $\mu_X(t) \pm 2\sigma_X(t)$ (solid lines). The latter curves have to be interpreted with care since the distributions of the $X_t$'s are not normal.

For $s \leq t$, $B_t - B_s$ and $B_s$ are independent, and $B_t - B_s \overset{d}{=} B_{t-s}$. Hence

$$c_X(t,s) = EX_t X_s - EX_t EX_s$$

(1.17)

$$= e^{\mu(t+s)} E\sigma^2(B_t + B_s) - e^{(\mu + 0.5 \sigma^2)(t+s)}$$

$$= e^{\mu(t+s)} E\sigma^2[(B_t - B_s) + 2B_s] - e^{(\mu + 0.5 \sigma^2)(t+s)}$$

$$= e^{\mu(t+s)} E\sigma^2(B_t - B_s) E\sigma^2 B_s - e^{(\mu + 0.5 \sigma^2)(t+s)}$$

$$= e^{(\mu + 0.5 \sigma^2)(t+s)} \left(e^{\sigma^2 s} - 1\right).$$

In particular, geometric Brownian motion has variance function

$$\sigma_X^2(t) = e^{(2\mu + \sigma^2)t} \left(e^{\sigma^2 t} - 1\right).$$

(1.18)

See Figure 1.3.9 for an illustration of various sample paths of geometric Brownian motion.
Example 1.3.10 (Gaussian white and colored noise)
In statistics and time series analysis one often uses the name “white noise” for a sequence of iid or uncorrelated random variables. This is in contrast to physics, where white noise is understood as a certain derivative of Brownian sample paths. This does not contradict our previous remarks since this derivative is not obtained by ordinary differentiation. Since white noise is “physically impossible”, one considers an approximation to it, called colored noise. It is a Gaussian process defined as

\[ X_t = \frac{B_{t+h} - B_t}{h}, \quad t \geq 0, \tag{1.19} \]

where \( h > 0 \) is some fixed constant. Its expectation and covariance functions are given by

\[ \mu_X(t) = 0 \quad \text{and} \quad c_X(t, s) = h^{-2}[(s + h) - \min(s + h, t)], \quad s \leq t. \]

Notice that \( c_X(t, s) = 0 \) if \( t - s \geq h \), hence \( X_t \) and \( X_s \) are independent, but if \( t - s < h \), \( c_X(t, s) = h^{-2}[h - (t - s)] \). Since \( X \) is Gaussian and \( c_X(t, s) \) is a function only of \( t - s \), it is stationary (see Example 1.2.8).

Clearly, if \( B \) was differentiable, we could let \( h \) in (1.19) go to zero, and in the limit we would obtain the ordinary derivative of \( B \) at \( t \). But, as we know, this argument is not applicable. The variance function \( \sigma_X^2(t) = h^{-1} \) gives an indication that the fluctuations of colored noise become larger as \( h \) decreases. Simulated paths of colored noise look very much like the sample paths in Figure 1.2.6.

\[ \square \]

1.3.3 Simulation of Brownian Sample Paths
This section is not necessary for the understanding of stochastic calculus. However, it will characterize Brownian motion as a distributional limit of partial sum processes (so-called functional central limit theorem). This observation will help you to understand the Brownian path properties (non-differentiability, unbounded variation) much better. A second objective of this section is to show that Brownian sample paths can easily be simulated by using standard software.

Using the almost unlimited power of modern computers, you can visualize the paths of almost every stochastic process. This is desirable because we like to see sample paths in order to understand the stochastic process better. On the other hand, simulations of the paths of stochastic processes are sometimes unavoidable if you want to say something about the distributional properties of such a process. In most cases, we cannot determine the exact distribution
4.1 The Black–Scholes Option Pricing Formula

4.1.1 A Short Excursion into Finance

We assume that the price \( X_t \) of a risky asset (called stock) at time \( t \) is given by geometric Brownian motion of the form

\[
X_t = f(t, B_t) = X_0 e^{(c-0.5\sigma^2)t + \sigma B_t},
\]

(4.1)

where, as usual, \( B = (B_t, t \geq 0) \) is Brownian motion, and \( X_0 \) is assumed to be independent of \( B \). The motivation for this assumption on \( X \) comes from the fact that \( X \) is the unique strong solution of the linear stochastic differential equation

\[
X_t = X_0 + c \int_0^t X_s \, ds + \sigma \int_0^t X_s \, dB_s,
\]

which we can formally write as

\[
dX_t = cX_t \, dt + \sigma X_t \, dB_t.
\]

This was proved in Example 3.2.4. If we interpret this equation in a naive way, we have on \([t, t + dt]\):

\[
X_{t+dt} - X_t = c X_t \, dt + \sigma X_t \, dB_t.
\]

Equivalently,

\[
\frac{X_{t+dt} - X_t}{X_t} = c \, dt + \sigma \, dB_t.
\]

The quantity on the left-hand side is the relative return from the asset in the period of time \([t, t + dt]\). It tells us that there is a linear trend \( c \, dt \) which is disturbed by a stochastic noise term \( \sigma \, dB_t \). The constant \( c > 0 \) is the so-called mean rate of return, and \( \sigma > 0 \) is the volatility. A glance at formula (4.1) tells us that, the larger \( \sigma \), the larger the fluctuations of \( X_t \). You can also check this with the formula for the variance function of geometric Brownian motion, which is provided in (1.18). Thus \( \sigma \) is a measure of the riskiness of the asset.

It is believed that the model (4.2) is a reasonable, though crude, first approximation to a real price process. If you forget for the moment the term with \( \sigma \), i.e. assume \( \sigma = 0 \), then (4.2) is a deterministic differential equation which has the well-known solution \( X_t = X_0 \exp\{ct\} \). Thus, if \( \sigma > 0 \), we should expect to obtain a randomly perturbed exponential function, and this is the geometric Brownian motion (4.1). People in economics believe in exponential growth, and therefore they are quite satisfied with this model.
4.1. THE BLACK–SCHOLES OPTION PRICING FORMULA

Now assume that you have a non-risky asset such as a bank account. In financial theory, it is called a bond. We assume that an investment of \( \beta_0 \) in bond yields an amount of

\[
\beta_t = \beta_0 e^{rt}
\]

at time \( t \). Thus your initial capital \( \beta_0 \) has been continuously compounded with a constant interest rate \( r > 0 \). This is an idealization since the interest rate changes with time as well. Note that \( \beta \) satisfies the deterministic integral equation

\[
\beta_t = \beta_0 + r \int_0^t \beta_s \, ds.
\]  

(4.3)

In general, you want to hold certain amounts of shares: \( a_t \) in stock and \( b_t \) in bond. They constitute your portfolio. We assume that \( a_t \) and \( b_t \) are stochastic processes adapted to Brownian motion and call the pair

\[
(a_t, b_t), \quad t \in [0, T],
\]

a trading strategy. Clearly, you want to choose a strategy, where you do not lose. How to choose \( (a_t, b_t) \) in a reasonable way, will be discussed below. Notice that your wealth \( V_t \) (or the value of your portfolio) at time \( t \) is now given by

\[
V_t = a_t X_t + b_t \beta_t.
\]

We allow both, \( a_t \) and \( b_t \), to assume any positive or negative values. A negative value of \( a_t \) means short sale of stock, i.e. you sell the stock at time \( t \). A negative value of \( b_t \) means that you borrow money at the bond's riskless interest rate \( r \). In reality, you would have to pay transaction costs for operations on stock and sale, but we neglect them here for simplicity. Moreover, we do not assume that \( a_t \) and \( b_t \) are bounded. So, in principle, you should have a potentially infinite amount of capital, and you should allow for unbounded debts as well. Clearly, this is a simplification, which makes our mathematical problems easier. And finally, we assume that you spend no money on other purposes, i.e. you do not make your portfolio smaller by consumption.

We assume that your trading strategy \( (a_t, b_t) \) is self-financing. This means that the increments of your wealth \( V_t \) result only from changes of the prices \( X_t \) and \( \beta_t \) of your assets. We formulate the self-financing condition in terms of differentials:

\[
dV_t = d(a_t X_t + b_t \beta_t) = a_t \, dX_t + b_t \, d\beta_t,
\]

which we interpret in the Itô sense as the relation

\[
V_t - V_0 = \int_0^t d(a_s \, X_s + b_s \, \beta_s) = \int_0^t a_s \, dX_s + \int_0^t b_s \, d\beta_s.
\]
The integrals on the right-hand side clearly make sense if you replace $dX_s$ with $cX_s \, ds + \sigma X_s \, dB_s$, see (4.2), and $d\beta_s$ with $r \beta_s \, ds$, see (4.3). Hence the value $V_t$ of your portfolio at time $t$ is precisely equal to the initial investment $V_0$ plus capital gains from stock and bond up to time $t$.

### 4.1.2 What is an Option?

Now suppose you purchase a ticket, called an option, at time $t = 0$ which entitles you to buy one share of stock until or at time $T$, the time of maturity or time of expiration of the option. If you can exercise this option at a fixed price $K$, called the exercise price or strike price of the option, only at time of maturity $T$, this is called a European call option. If you can exercise it until or at time $T$, it is called an American call option. Note that there are many more different kinds of options in the real world of finance but we will not be able to include them in this book.

The holder of a call option is not obliged to exercise it. Thus, if at time $T$ the price $X_T$ is less than $K$, the holder of the ticket would be silly to exercise it (you could buy one share for $X_T$ on the market!), and so the ticket expires as a worthless contract. If the price $X_T$ exceeds $K$, it is worthwhile to exercise the call, i.e. one buys the share at the price $K$, then turns around and sells it at the price $X_T$ for a net profit $X_T - K$.

In sum, the purchaser of a European call option is entitled to a payment of

$$(X_T - K)^+ = \max(0, X_T - K) = \begin{cases} X_T - K, & \text{if } X_T > K, \\ 0, & \text{if } X_T \leq K. \end{cases}$$

See Figure 4.1.1 for an illustration.

A put is an option to sell stock at a given price $K$ on or until a particular date of maturity $T$. A European put option is exercised only at time of maturity, an American put can be exercised until or at time $T$. The purchaser of a European put makes profit

$$(K - X_T)^+ = \begin{cases} K - X_T, & \text{if } X_T < K, \\ 0, & \text{if } X_T \geq K. \end{cases}$$

In our theoretical considerations we restrict ourselves to European calls. This has a simple reason: in this case we can derive explicit solutions and compact formulae for our pricing problems. Thus,

from now on, an option is a European call option.
As an aside, it is interesting to note that the situation can be imagined colorfully as a game where the reward is the payoff of the option and the option holder pays a fee (the option price) for playing the game.

Since you do not know the price $X_T$ at time $t = 0$, when you purchase the call, a natural question arises:

How much would you be willing to pay for such a ticket, i.e. what is a rational price for this option at time $t = 0$?

Black, Scholes and Merton defined such a value as follows:

- An individual, after investing this rational value of money in stock and bond at time $t = 0$, can manage his/her portfolio according to a self-financing strategy (see p. 169) so as to yield the same payoff $(X_T - K)^+$ as if the option had been purchased.

- If the option were offered at any price other than this rational value, there would be an opportunity of arbitrage, i.e. for unbounded profits without an accompanying risk of loss.
4.1.3 A Mathematical Formulation of the Option Pricing Problem

Now suppose we want to find a self-financing strategy \((a_t, b_t)\) and an associated value process \(V_t\) such that

\[
V_t = a_t X_t + b_t \beta_t = u(T - t, X_t), \quad t \in [0, T],
\]

for some smooth deterministic function \(u(t, x)\). Clearly, this is a restriction: you assume that the value \(V_t\) of your portfolio depends in a smooth way on \(t\) and \(X_t\). It is our aim to find this function \(u(t, x)\). Since the value \(V_T\) of the portfolio at time of maturity \(T\) shall be \((X_T - K)^+\), we get the *terminal condition*

\[
V_T = u(0, X_T) = (X_T - K)^+.
\]  \(\quad (4.4)\)

In the financial literature, the process of building a self-financing strategy such that (4.4) holds is called *hedging against the contingent claim \((X_T - K)^+\).*

We intend to apply the Itô lemma to the value process \(V_t = u(T - t, X_t)\). Write \(f(t, x) = u(T - t, x)\) and notice that

\[
f_1(t, x) = -u_1(T - t, x), \quad f_2(t, x) = u_2(T - t, x), \quad f_{22}(t, x) = u_{22}(T - t, x).
\]

Also recall that \(X\) satisfies the Itô integral equation

\[
X_t = X_0 + c \int_0^t X_s ds + \sigma \int_0^t X_s dB_s.
\]

Now an application of the Itô lemma (2.30) with \(A^{(1)} = c X\) and \(A^{(2)} = \sigma X\) yields that

\[
V_t - V_0 = f(t, X_t) - f(0, X_0)
\]

\[
= \int_0^t [f_1(s, X_s) + c X_s f_2(s, X_s) + 0.5 \sigma^2 X_s^2 f_{22}(s, X_s)] ds
\]

\[
+ \int_0^t [\sigma X_s f_2(s, X_s)] dB_s
\]

\[
= \int_0^t [-u_1(T - s, X_s) + c X_s u_2(T - s, X_s)
\]

\[
+ 0.5 \sigma^2 X_s^2 u_{22}(T - s, X_s)] ds
\]

\[
+ \int_0^t [\sigma X_s u_2(T - s, X_s)] dB_s.
\]  \(\quad (4.5)\)
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On the other hand, \((a_t, b_t)\) is self-financing:

\[
V_t - V_0 = \int_0^t a_s \, dX_s + \int_0^t b_s \, d\beta_s. \tag{4.6}
\]

Since \(\beta_t = \beta_0 e^{rt}\),

\[
d\beta_t = r \beta_0 e^{rt} \, dt = r \beta_t \, dt. \tag{4.7}
\]

Moreover, \(V_t = a_t X_t + b_t \beta_t\), thus

\[
b_t = \frac{V_t - a_t X_t}{\beta_t}. \tag{4.8}
\]

Combining (4.6)–(4.8), we obtain another expression for

\[
V_t - V_0 = \int_0^t a_s \, dX_s + \int_0^t \frac{V_s - a_s X_s}{\beta_s} \, r \beta_s \, ds
= \int_0^t a_s \, dX_s + \int_0^t r (V_s - a_s X_s) \, ds
= \int_0^t c a_s X_s \, ds + \int_0^t \sigma a_s X_s \, dB_s + \int_0^t r (V_s - a_s X_s) \, ds
= \int_0^t [(c - r) a_s X_s + r V_s] \, ds + \int_0^t [\sigma a_s X_s] \, dB_s. \tag{4.9}
\]

Now compare formulae (4.5) and (4.9). We learnt on p. 119 that coefficient functions of Itô processes coincide. Thus we may formally identify the integrands of the Riemann and Itô integrals, respectively, in (4.5) and (4.9):

\[
a_t = u_2(T - t, X_t), \tag{4.10}
\]

\[
(c - r) a_t X_t + r u(T - t, X_t) = (c - r) u_2(T - t, X_t) \, X_t + r u(T - t, X_t)
= -u_1(T - t, X_t) + c X_t u_2(T - t, X_t)
+ 0.5 \sigma^2 X_t^2 u_22(T - t, X_t). \tag{4.5}
\]

Since \(X_t\) may assume any positive value, we can write the last identity as a partial differential equation ("partial" refers to the use of the partial derivatives of \(u\)):

\[
u_1(t, x) = 0.5 \sigma^2 x^2 u_22(t, x) + r x u_2(t, x) - r u(t, x), \tag{4.11}
\]

\(x > 0\), \(t \in [0, T]\).
Recalling the terminal condition (4.4), we also require that
\[ V_T = u(0, X_T) = (X_T - K)^+ . \]
This yields the deterministic terminal condition
\[ u(0, x) = (x - K)^+, \quad x > 0 . \] (4.12)

4.1.4 The Black and Scholes Formula

In general, it is hard to solve a partial differential equation explicitly, and so one has to rely on numerical solutions. So it is somewhat surprising that the partial differential equation (4.11) has an explicit solution. (This is perhaps one of the reasons for the popularity of the Black–Scholes–Merton approach.) The partial differential equation (4.11) with terminal condition (4.12) has been well-studied; see for example Zauderer (1989). It has the explicit solution
\[ u(t, x) = x \Phi(g(t, x)) - K e^{-r t} \Phi(h(t, x)) , \]
where
\[ g(t, x) = \frac{\ln(x/K) + (r + 0.5 \sigma^2) t}{\sigma t^{1/2}} , \]
\[ h(t, x) = g(t, x) - \sigma t^{1/2} , \]
and
\[ \Phi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-y^2/2} dy , \quad x \in \mathbb{R} , \]
is the standard normal distribution function.

After all these calculations,

what did we actually gain?

Recalling our starting point on p. 171, we see that
\[ V_0 = u(T, X_0) = X_0 \Phi(g(T, X_0)) - K e^{-r T} \Phi(h(T, X_0)) \] (4.13)
is a rational price at time \( t = 0 \) for a European call option with exercise price \( K \).

The stochastic process \( V_t = u(T-t, X_t) \) is the value of your self-financing portfolio at time \( t \in [0, T] \).
At time of maturity $T$, the formula (4.13) yields the net portfolio value of $(X_T - K)^+$. Moreover, one can show that $a_t > 0$ for all $t \in [0, T]$, but $b_t < 0$ is not excluded. Thus short sales of stock do not occur, but borrowing money at the bond’s constant interest rate $r > 0$ may become necessary.

Equation (4.13) is the celebrated Black-Scholes option pricing formula. We see that it is independent of the mean rate of return $c$ for the price $X_t$, but it depends on the volatility $\sigma$.

If we want to understand $q = u(T, X_0)$ as a rational value in terms of arbitrage, suppose that the initial option price $p \neq q$. If $p > q$, apply the following strategy: at time $t = 0$

- sell the option to someone else at the price $p$, and
- invest $q$ in stock and bond according to the self-financing strategy (4.14).

Thus you gain an initial net profit of $p - q > 0$. At time of maturity $T$, the portfolio has value $a_T X_T + b_T \beta_T = (X_T - K)^+$, and you have the obligation to pay the value $(X_T - K)^+$ to the purchaser of the option. This means: if $X_T > K$, you must buy the stock for $X_T$, and sell it to the option holder at the exercise price $K$, for a net loss of $X_T - K$. If $X_T \leq K$, you do not have to pay anything, since the option will not be exercised. Thus the total terminal profit is zero, and the net profit is $p - q$.

The scale of this game can be increased arbitrarily, by selling $n$ options for $np$ at time zero and by investing $nq$ in stock and bond according to the self-financing strategy $(n a_t, n b_t)$. The net profit will be $n (p - q)$. Thus the opportunity for arbitrarily large profits exists without an accompanying risk of loss. This means arbitrage. Similar arguments apply if $q > p$; now the purchaser of the option will make arbitrarily big net profits without accompanying risks.

Notes and Comments

The idea of using Brownian motion in finance goes back to Bachelier (1900), but only after 1973, when Black, Scholes and Merton published their papers,
did the theory reach a more advanced level. Since then, options, futures and many other financial derivatives have conquered the international world of finance. This led to a new, applied, dimension of an advanced mathematical theory: stochastic calculus. As we have learnt in this book, this theory requires some non-trivial mathematical tools.

In 1997, Merton and Scholes were awarded the Nobel prize for economics. Most books, which are devoted to mathematical finance, require the knowledge of measure theory and functional analysis. For this reason they can be read only after several years of university education! Here are a few references: Duffie (1996), Musiela and Rutkowski (1997) and Karatzas and Shreve (1998), and also the chapter about finance in Karatzas and Shreve (1988).

By now, there also exist a few texts on mathematical finance which address an elementary or intermediate level. Baxter and Rennie (1996) is an easy introduction with a minimum of mathematics, but still precise and with a good explanation of the economic background. The book by Willmot, Howison and Dewyne (1995) focuses on partial differential equations and avoids stochastic calculus whenever possible. A course on finance and stochastic calculus is given by Lamberton and Lapeyre (1996) who address an intermediate level based on some knowledge of measure theory. Pliska (1997) is an introduction to finance using only discrete-time models.

4.2 A Useful Technique: Change of Measure

In this section we consider a very powerful technique of stochastic calculus: the change of the underlying probability measure. In the literature it often appears under the synonym Girsanov's theorem or Cameron–Martin formula.

In what follows, we cannot completely avoid measure-theoretic arguments. If you do not have the necessary background on measure theory, you should at least try to understand the main idea by considering the applications in Section 4.2.2.

4.2.1 What is a Change of the Underlying Measure?

The main idea of the change of measure technique consists of introducing a new probability measure via a so-called density function which is in general not a probability density function.

We start with a simple example of two distributions on the real line. Recall