Geometric Distribution

Geometric Dist. is concerned with the number of trials that must be conducted until a specific outcome is observed.

Ex: Toss thumbtack (repeatedly)

Thumbtack can land with the spindle pointing up (U) or with spindle pointing down (D).

Suppose $P(D) = p$. So $P(U) = 1 - p$. We will toss the thumbtack repeatedly until
until the first time the tack lands with the spindle down (D).

a) What is the Sample Space?
   \[ SS = \{ D, UD, UUD, UUUD, \ldots \} \]

b) Let r.v. \( X \) count the number of tosses. Construct the prob. dist. for \( X \).

\[
P(X=1) = P(D) = p
\]

\[
P(X=2) = P(UD) = P(U) \times P(D) = (1-p)p
\]

\[
P(X=3) = P(UUD) = P(U) \times P(U) \times P(D) = (1-p)(1-p)p
\]
\[ P(X=4) = P(\text{uuuud}) \]
\[ = P(u) P(u) P(u) P(d) \]
\[ = (1-p)(1-p)(1-p)p \]

Generalizing, we can get the probability density function (pdf) of \( X \):

\[ P(X=x) = (1-p)^{x-1}p \]

for \( x=1,2,3,4, \ldots \)

**Geometric Distribution:**

Suppose an experiment

1) consists of a sequence of independent trials
2) Each trial results in one of only two possible outcomes (Success or Failure).

3) \( p = P(\text{Success}) \) is the same on each trial.

4) r.v. \( X \) counts the number of trials it takes to observe the first Success.

Then \( X \) has a geometric dist. with

\[
P(X = x) = (1 - p)^{x-1} p \quad \text{for } x = 1, 2, \ldots
\]

Also

\[
\mu_X = \frac{1}{p} \quad \sigma_X = \sqrt{\frac{1 - p}{p^2}}
\]
EX: Toss fair coin until $H$ appears. $X$ counts the number of tosses.

a) Find prob. that $H$ appears on first toss

$$P(X=1) = (1-.5)^{1-1} \cdot (.5)$$
$$= (.5^0) \cdot (.5)$$
$$= 1 \cdot (.5)$$
$$= .5$$

b) Find prob. that first $H$ appears on second toss.

$$P(X=2) = (1-.5)^{2-1} \cdot (.5)$$
$$= (.5)^1 \cdot (.5)$$
$$= .25$$
c) Find prob. that first H appears on 5th toss.

\[ P(X=5) = (1-.5)^{5-1} \cdot (.5) \]

\[ = (.5)^4 \cdot (.5) \]

\[ = (.0625)(.5) \]

\[ = .03125 \]

d) What is the expected number of tosses for the first H to appear?

\[ \mu_X = \frac{1}{p} = \frac{1}{.5} = 2 = E(X) \]
Using similar reasoning, we can find prob. that it takes more than \( x \) trials for the first success to appear:

\[
P(X > x) = P(\text{first } x \text{ trials result in failure})
\]

\[
= P(\text{first trial Failure}) \times \\
 \times P(\text{second trial Failure}) \times \\
 \vdots \times \\
 \times P(\text{\( x^{th} \) trial Failure}) \\
= (1-p)^x
\]

Ex: com exemple (cont.)
e) Find prob. that more than 4 tosses will be required to get the first H.

\[ P(X > 4) = (1 - .5)^4 \\
= (.5)^4 \]

\[ = .0625 \]

f) Find prob. that first H appears before the 6th toss

\[ P(X < 6) = P(X \leq 5) \]

\[ = 1 - P(X > 5) \]

\[ = 1 - (1 - .5)^5 \]

\[ = 1 - (0.5)^5 \]

\[ = 1 - 0.03125 \]

\[ = 0.96875 \]
g) Find prob. that no more than 5 tosses are required to observe the first H

\[ P(X \leq 5) = 1 - P(X > 5) \]
\[ = 1 - (0.5)^5 \]
\[ = 0.96875 \]

h) Find prob. that at least 3 tosses will be required to get first H.

"At least 3" means "3 or more"

\[ P(X \geq 3) = P(X > 2) \]
\[ = (1 - 0.5)^2 \]
\[ = (0.5)^2 \]
\[ = 0.25 \]