1. Probit and Inverse CDF link functions

2. Poisson GLM of Independence in $1 \times J$ Contingency Tables

3. Exponential dispersion family
   
   3a. Mean
   
   3b. Variance
   
   3c. Examples: Poisson, Binomial

Homework: Ch. 4, Sec. 2, 3
Problems: 4.1, 4.3, 4.7
Consider logistic regression model with $\beta > 0$.

\[ \pi(x) = \frac{\exp(d + \beta x)}{1 + \exp(d + \beta x)} \quad (\star) \]

\[ = \Phi(d + \beta x) \]

where

\[ \Phi(y) = \frac{e^y}{1 + e^y} \] is a standard logistic CDF.

with the mean $\mu = 0$

variance $\sigma^2 = \pi^2 / 3$

($\pi = 3.14$).

From ($\star$), the link function can be specified by the inversion \( \Phi^{-1} \)

\[ \Phi^{-1}[\pi(x)] = d + \beta x. \]

\[ \Phi^{-1} : [0,1] \to (-\infty, +\infty) \]

This model is known as Probit model.
2. Poisson GLM of Independence in $1 \times 3$ Contingency Tables

Consider Two-way Table with independent counts

$Y_{ij} \sim \text{Poisson}(\mu_{ij})$

Assumptions: $\sum_i x_i = 1 \quad \sum_j \beta_j = 1$

Null Hypothesis: $H_0: \mu_{ij} = \mu \cdot x_i \cdot \beta_j$

This is multiplicative model.

The linear predictor for a GLM Poisson model is based on the log link function:

$$\log(\mu_{ij}) = \alpha + \alpha_i^* + \beta_j^*,$$

where $\alpha = \log \mu$

$\alpha_i^* = \log \alpha_i$

$\beta_j^* = \log \beta_j$
i.e. the log linear model for Poisson counts has additive effects of the two classifications but no interaction.

Connection with the multinomial model:

\[ Y_{ij} \text{ are independent } \sim \text{ Poisson} \left( \mu_{ij} \right) \]

\[ \sum_{i} Y_{ij} \sim \text{ Poisson} \left( \sum_{i} \mu_{ij} \right) \]

but under the model

\[ H_0 : \mu_{ij} = \lambda_i \beta_j \]

\[ \sum_{i} \mu_{ij} = \mu \]

so that

\[ Y_{ij} \mid \sum_{i} Y_{ij} = n \sim \text{ multinomial} \left( i, \pi_{ij} \right) \]

\[ \pi_{ij} = \frac{\mu_{ij}}{\mu} \text{ i.e. } \pi_{ij} = \lambda_i \beta_j \]
Also: for each fixed $i$:

$$\sum_{j} Y_{ij} \mid \sum Y_{ij} = n \sim \text{Bin}(\mu, \pi_{i+})$$

where

$$\pi_{i+} = \frac{\sum \mu_{ij}}{\mu}$$

$$= \frac{\mu (\sum B_{j})}{\mu} = B_{i}$$

and for each fixed $j$:

$$\sum_{i} Y_{ij} \mid \sum Y_{ij} = n \sim \text{Bin}(\mu, \pi_{+j})$$

where $\pi_{+j} = B_{j}$

i.e. conditional on $\sum Y_{ij} = n$

the model is multinomial one

with

$$\pi_{ij} = \pi_{i+} \cdot \pi_{+j}$$

$$= \pi_{i+} \cdot \pi_{+j}$$

this is independence of the two classifications.
3. Exponential dispersion family:

Assume that there are two parameters under the model (mean & variance).

The random component for this type of GLM model is specified by

\[ Y_i = f(y_i, \theta_i, \phi) = \]

\[ = \exp \left[ \frac{(y_i - \theta_i)}{\phi} \right] + c(y_i, \phi) \]

-is the so-called exponential dispersion family and \( \phi \) is called the dispersion parameter.

-the parameter \( \theta_i \) is the natural parameter.

When \( \phi \) is known the model (1) is usual GLM model.
with

\[ f(y_i; \theta; \phi) = a(\theta; \phi) b(y_i) \exp(y_i; Q(\theta; \phi)) \]

where

\[ a(\theta) = \exp \left( -b(\theta) / a(\phi) \right) \]

\[ b(y) = \exp(-c(y, \phi)) \]

\[ Q(\theta) = \frac{\theta}{a(\phi)} \]

Remark #1. Usually \( a(\phi) = \frac{\phi}{\omega_i} \)

where \( \omega_i \) are known weights.

Log likelihood function:

\[ \sum_i L_i \]

where

\[ L_i = \log f(y_i; \theta; \phi) = \frac{y_i; \theta; \phi - b(\theta)}{a(\phi)} + c(y_i; \phi) \]

\[ \frac{\partial L_i}{\partial \theta_i} = \frac{y_i - b'(\theta_i)}{a(\phi)} ; \quad \sum_i \frac{\partial L_i}{\partial \theta_i} = 0 \]

\[ \frac{\partial^2 L_i}{\partial \theta_i^2} = -\frac{b''(\theta_i)}{a(\phi)} ; \quad \mathbb{E}\left( \frac{\partial^2 L}{\partial \theta^2} \right) = \mathbb{E}\left[ \frac{\partial^2 L}{\partial \theta^2} \right] \]
So that we have, respectively the following expressions for mean and variance of \( Y_i \):

\[
\frac{\mathbb{E} \left( Y_i - b'(\Theta_i) \right)}{a(\phi)} = 0, \quad \text{i.e.} \quad \mathbb{E}(Y_i) = b'(\Theta_i)
\]

\[
\frac{b''(\Theta_i)}{a(\phi)} = -\mathbb{E} \left( \frac{(Y_i - b(\Theta_i))^2}{a(\phi)} \right)
\]

\[
b''(\Theta_i) a(\phi) = \mathbb{E} (Y_i - \mu_i)^2 = \text{var}(Y_i)
\]

Remark #3. The function \( b(\cdot) \) determines moments of \( Y_i \).
Mean & Variance of Poisson

Random component

Assume that \( Y_i \sim \text{Poisson} (\mu_i) \)

\[
f(y_i, \mu_i) = \frac{\mu_i^{y_i} e^{-\mu_i}}{(y_i)!} = \\
= \exp \left\{ y_i \log \mu_i - \mu_i - \log (y_i)! \right\} \\
= \exp \left\{ y_i \Theta_i - \exp(\Theta_i) - \log (y_i)! \right\} \\
\text{where } \Theta_i = \log \mu_i \\
\mu_i = \exp(\Theta_i)
\]

i.e. \( Y \) has exponential dispersion

form \((1)\) with \( b(\Theta) = \exp(\Theta) \)

\( a(\varphi) = 1 \)

\( c(y, \varphi) = -\log (y)! \)

The natural parameter

\( \Theta = \log \mu \)
mean of $Y_i$: $E(Y_i) = B'(\theta_i) = \frac{\exp(\theta_i)}{[\exp(\theta_i)]'} = \exp(\theta_i) = \mu_i$

covariance of $Y_i$: $\text{var}(Y_i) = B''(\theta_i) = (\exp(\theta_i))'' = \exp(\theta_i) = \mu_i$

**Mean and variance for Binomial random component:**

Let $Y_i$ be a sample proportion in $n_i$ Bernoulli trials.

namely $Y_i \sim \text{Bin}(\cdot, n_i, \pi_i)$

Let $\theta_i = \log \frac{\pi_i}{1-\pi_i}$

then $\pi_i = \frac{\exp(\theta_i)}{1 + \exp(\theta_i)}$ and $\log(1-\pi_i) = -\log \left[1 + \exp(\theta_i)\right]$
Note also, that
\[ n_i y_i \sim \text{Bin}(\cdot, n_i, \pi_i), \]
and
\[ f(y_i; \pi_i, n_i) = \frac{(n_i)^{n_i y_i} (n_i - n_i y_i)^{n_i - n_i y_i}}{n_i^{y_i}} \pi_i^{y_i} (1 - \pi_i)^{n_i - y_i} = \]
\[ = \exp \left[ y_i \theta_i - \log(1 + \exp(\theta_i)) \right] + \log \left( \frac{\pi_i}{1 - \pi_i} \right) \]
i.e. we have exponential dispersion family with
\[ f(\theta) = \log(1 + \exp(\theta)) \]
\[ a(\phi) = \frac{1}{n_i} \left( w_i = n_i \right) \]
\[ c(y_i; \phi) = \log \left( \frac{n_i}{n_i y_i} \right), \]
and the natural parameter
\[ \Theta_i = \log \left( \frac{\pi_i}{1 - \pi_i} \right). \]
In this model

\[ E(Y_i) = \beta'(\theta_i) = \frac{\exp(\theta_i)}{1 + \exp(\theta_i)} = \pi_i \]

\[ \text{var}(Y_i) = \beta''(\theta) \cdot \alpha(\theta) = \]

\[ = \frac{\exp(\theta_i) \left[ \frac{1 + \exp(\theta_i)}{1 + \exp(\theta_i)} \right] - \exp(\theta_i) \cdot \exp(\theta_i) \cdot \alpha(\theta)}{\left[1 + \exp(\theta_i)\right]^2} \]

\[ = \frac{\exp(\theta_i) \cdot 1}{1 + \exp(\theta_i)} \cdot \frac{1}{1 + \exp(\theta_i)} \cdot \frac{1}{n_i} = \]

\[ = \pi_i \left(1 - \pi_i\right) \frac{1}{n_i} \]