Problem #1. A manufacturer produces cars that have a length of life that is normally distributed with mean $\mu = 7$ years and standard deviation $\sigma = 2$ years. What is the probability that a car will last between 3 and 6 years?

Problem #2. Let $\bar{X}$ denote the mean of a random sample of size $n=49$ from a distribution with mean $\mu=0.5$ and standard deviation $\sigma=0.3$. Approximate the probability $P(0.4 < \bar{X} < 0.6)$. State the name of the theorem you are using.

Problem #3. A multiple choice test has 72 questions, with each question having three possible choices. If a student selects his answers at random calculate the probability, by using the normal approximation, that he will obtain at least 32 correct answers.

Problem #4. The mean and standard deviation of $n=10$ SAT test scores selected at random from the entering freshman class are $\bar{X}=1100$ and $s=125$.

a) State your assumptions about the distribution of SAT test scores to find confidence interval for the population mean $\mu$.

b) Find the 95% confidence interval for the population mean $\mu$.

Problem #5. In comparing the mean weight loss for two diets the following sample data were obtained:

<table>
<thead>
<tr>
<th>Diet</th>
<th>Sample size</th>
<th>Sample Mean</th>
<th>Sample Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diet I</td>
<td>10</td>
<td>9.21</td>
<td>4.3</td>
</tr>
<tr>
<td>Diet II</td>
<td>10</td>
<td>7.88</td>
<td>5.7</td>
</tr>
</tbody>
</table>

Assuming independent normal distributions with equal variances for weight loss with diets I and II, construct a 95% confidence interval for $\mu_1 - \mu_2$.

Problem #6. On a certain day, a large number of fuses were manufactured, each rated at 15A. A sample of 75 fuses is drawn from the day’s production, and 17 of them were found to have burnout amperages greater than 15A. Find a 95% confidence interval for the proportion of fuses manufactured that day whose burnout amperage is greater than 15A.

Problem #7. In a study of the flammability of material used in children’s sleepwear, the char length (inches) for five samples of washed acetate/nylon brushed tricot fabric was measured. The resulting observations were

8.1  10.4  9.5  8.9  10.7

Assuming that char length is normally distributed variable, compute a 95% CI for the variance $\sigma^2$. 

Problem #8. Find the mean, the variance, 50-th percentile (median), the 25-th percentile (first quartile), and the 75-th percentile (third quartile), for random variable X with the probability density function (pdf)
\[ f(x) = 2x, \quad 0 \leq x \leq 1. \]

Problem #9. What is the mean and variance of the random variable \( U = 18^{1/2} \left( X - \frac{2}{3} \right) \)? Here the random variable \( X \) is defined in Problem #8.

*) Problems #1–8 are worth 20 points each. Problem #9 is optional and worth extra 5 points.
Problem #1: Let $X$ = the length of life of a car
$X \sim \text{Normal} (\mu = 7, \sigma = 2)$
$$P(3 < X < 6) = P\left(\frac{3-7}{2} < Z < \frac{6-7}{2}\right) =$$
$$= P(-2 < Z < -0.5) = \Phi(-0.5) - \Phi(-2) =$$
$$= 1 - \Phi(0.5) - (1 - \Phi(2)) = 1 - 0.6915 - 1 + 0.9772 =$$
$$= 0.2857$$

Problem #2: $n = 49$, CLT states $\bar{X} \sim N(0.5, \frac{3^2}{49})$
$X_1, \ldots, X_{49}$ are i.i.d. of any pdf.
$$Z = \frac{\bar{X} - 0.5}{\frac{3}{7}}$$
$$P(0.4 < \bar{X} < 0.6) = P\left(\frac{0.4 - 0.5}{\frac{3}{7}} < Z < \frac{0.6 - 0.5}{\frac{3}{7}}\right) =$$
$$= P(-2.33 < Z < 2.33) = \Phi(2.33) - \Phi(-2.33) =$$
$$= 2 \Phi(2.33) - 1 = 2 \times (0.9901) - 1 = 0.9802$$
Problem #3: \( n = 72 \) questions

Correct with \( p = \frac{1}{3} \)

Correct with \( q = \frac{2}{3} \)

Let \( X \) = the number of correct answers

\[ X = \text{Bin} (n=72, p=\frac{1}{3}) \]

Conditions for approx. \( \text{Bin} (n, p) \) by \( \text{Normal}(np, npq) \):

\[ np > 5 \text{ and } nq > 5 \]

\[ np = 72 \times \frac{1}{3} = 24 > 5 \]

\[ nq = 72 \times \frac{2}{3} = 48 > 5 \]

\[ Z = \frac{X - 24}{\sqrt{72 \times \frac{1}{3} \times \frac{2}{3}}} \]

\[ i.e. \ Z = \frac{X - 24}{4} \]

\[ P(X \geq 32) 
\approx P\left(Z \geq \frac{32 - 24}{4}\right) 
= P(Z \geq 2) 
= 1 - \Phi(2) 
= 1 - .9772 = .0228 \]

Problem #4. \( n = 10 \) SAT scores selected at random

\[ X_1, \ldots, X_{100} \text{, with } \bar{X} = 1100, S = 125 \]

(a) Assumption: \( X_i \sim \text{Normal}(\mu, \sigma^2) \)

(b) 95\% CI \( \mu \):

\[ \bar{X} \pm t_{0.025, 9 \times 10} \times \frac{S}{\sqrt{n}} \]

\[ = 1010.59 \leq \mu \leq 1189.41 \]
Problem #5: \( n_1 = 10 < 30 \quad \bar{X} = 9.21; \quad s^2 = 4.3 \) 
\( n_2 = 10 < 30 \quad \bar{Y} = 7.88; \quad s^2 = 5.7 \)

Assumptions:

\( X_i \sim \text{Normal}(\mu_1, \sigma_i^2) \)
\( Y_i \sim \text{Normal}(\mu_2, \sigma_i^2) \) and \( \sigma_1^2 = \sigma_2^2 = \sigma^2 \)

95% CI for \( \mu_1 - \mu_2 \): \( \bar{X} - \bar{Y} + t_{0.025, 18} \times \sqrt{s_p^2 \left( \frac{1}{10} + \frac{1}{10} \right)} \)

with \( s_p^2 = \frac{9.43 + 9.57}{10 + 10 - 2} = 5 \) - is a pooled estimator of \( \sigma^2 \).

i.e.,

\( \bar{X} - \bar{Y} + t_{0.025, 18} \times \sqrt{5 \left( \frac{1}{10} + \frac{1}{10} \right)} \)

1.33 + 2.101

\[ 0.77 \leq \mu_1 - \mu_2 \leq 3.43 \]

Problem #6: \( n = 75 \) fuses \( (n > 30) \)

\( \bar{p} = \frac{17}{75} = 0.227 \quad 1 - \alpha = 0.95 \quad \alpha = 0.05 \quad \alpha/2 = 0.025 \)

95%

\( \text{CI}_p : \quad \bar{p} \pm z_{0.025} \sqrt{\bar{p}(1-\bar{p}) \frac{1}{75}} \)

\[ 0.227 \pm 1.96 \sqrt{0.227(1-0.227) \frac{1}{75}} \]

0.1322 \leq p \leq 0.3218
Problem #7: Data: 8.1; 10.4; 9.5; 8.9; 10.7

\( n = 5 \), Assumption \( X_i \sim \text{Normal} \left( \mu, \sigma^2 \right) \), \( i = 1, 2, \ldots, 5 \).

95\% CI for \( \sigma^2 \):

\[
\frac{(n-1)s^2}{\chi^2_{.025,4}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{.975,4}}
\]

\[\chi^2_{.025,4} = 11.77 \]
\[\chi^2_{.975,4} = 0.484 \]

But

\[
X = \frac{1}{5} \left[ 8.1 + 10.4 + 9.5 + 8.9 + 10.7 \right] = 9.52
\]

\[
s^2 = \frac{1}{4} \left[ (8.1-9.52)^2 + (10.4-9.52)^2 + (9.5-9.52)^2 + (8.9-9.52)^2 + (10.7-9.52)^2 \right] = 1.142
\]

CI for \( \sigma^2 \):

\[
4 \times \frac{1.142}{11.77} \leq \sigma^2 \leq 4 \times \frac{1.142}{0.484}
\]

or

\[
4.10 \leq \sigma^2 \leq 9.438
\]

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Problem #8: Let \( X = f(x) = 2x, \ 0 \leq x \leq 1 \).

cdf \( F(x) = \int_0^x f(t) dt = \int_0^x 2t dt = x^2 \)

\( F(x_{.25}) = .25 \implies x^2 = .25 \implies x_{.25} = .5 \)

\( F(x_{.5}) = .5 \implies x^2 = .5 \implies x_{.5} = \sqrt{.5} \approx .707 \)

\( F(x_{.75}) = .75 \implies x^2 = .75 \implies x_{.75} = \sqrt{.75} \approx .866 \)

Mean of \( X \): \( E(X) = \int_0^1 x \cdot f(x) dx = \int_0^1 x \cdot 2x dx = 2 \int_0^1 x^2 dx = \frac{2}{3} \cdot x^3 \bigg|_0^1 = \frac{2}{3} \)

\( \text{Var}(X) = E(X^2) - (E(X))^2 = \int_0^1 x^2 \cdot f(x) dx - \left( \frac{2}{3} \right)^2 = \int_0^1 x^2 \cdot 2x dx - \frac{4}{9} = 2 \int_0^1 x^3 dx - \frac{4}{9} = 2 \cdot \frac{x^4}{4} \bigg|_0^1 - \frac{4}{9} = \frac{1}{18} \)

Problem #9: Let \( U = \sqrt{18} \left( X - \frac{2}{3} \right) \)

\( U = \frac{X - \frac{2}{3}}{\sqrt{\frac{1}{18}}} \) is a standardized r.v. (of \( X \))

\( E(U) = 0 \) \( \text{and} \) \( \text{Var}(U) = 1 \)

Indeed:

\( E(U) = \sqrt{18} \cdot E(X) - \sqrt{18} \cdot \frac{2}{3} = \sqrt{18} \cdot \frac{2}{3} - \sqrt{18} \cdot \frac{2}{3} = 0 \)

\( \text{Var}(U) = (\sqrt{18})^2 \cdot \text{Var}(X - \frac{2}{3}) = 18 \cdot \text{Var}(X) = 18 \cdot \frac{1}{18} = 1 \)