Problem #8. Find the 50-th percentile (median), the 25-th percentile (first quartile), the 75-th percentile (third quartile), for continuous distribution with the probability density function
\[ f(x) = 3x^2, \ 0 \leq x \leq 1. \]

Problem #9. The air pressure in a randomly selected tire put on a certain model new car is normally distributed with mean value 31 psi and standard deviation 0.2 psi.
(a) What is the probability that the pressure for a randomly selected tire exceeds 30.5 psi?
(b) What is the probability that the pressure for a randomly selected tire is between 30.5 and 31.5 psi?
(c) Suppose a tire is classified as under inflated if its pressure is less than 30.4 psi. What is the probability that at least one of the 4 tires on a car is under inflated?

Problem #10. Let \( X_1, X_2 \) be a random sample of size \( n=2 \) from the continuous pdf \( f(x) = 2x, \ 0 \leq x \leq 1 \). Let \( Y = X_1 + X_2 \).
(a) Compute the mean \( \mu_X \) and variance \( \sigma_X^2 \) of the underlying distribution and use them to determine \( \mu_Y \) and \( \sigma_Y^2 \).
(b) What is the mean and variance of the random variable \( U = 18^{1/2} (X_1 - 2/3) \)?
Solutions:

Problem #8: \( X \sim f(x) \) with

\[ f(x) = 3x^2, \quad \text{for} \quad 0 < x < 1. \]

CDF of \( X \):

\[ F(x) = \int_0^x 3t^2 \, dt = x^3 \]

The first quartile:

\[ x_{.25} : \quad F(x_{.25}) = .25 \]

\[ x_{.25} = (0.25)^{\frac{1}{3}} = 0.631 \]

The median \( x_{.5} \):

\[ F(x_{.5}) = .5 \]

\[ x_{.5} = (.5)^{\frac{1}{3}} = 0.794 \]

The third quartile:

\[ x_{.75} : \quad F(x_{.75}) = .75 \]

\[ x_{.75} = (.75)^{\frac{1}{3}} = 0.91 \]
The mean of \( X \):  
\[
\mu = E(X) = \int_0^1 x f(x) \, dx
\]
\[
\mu = \int_0^1 x \cdot 3x^2 \, dx = 3 \int_0^1 x^3 \, dx = \frac{3}{4} \left. x^4 \right|_{x=0}^{x=1} = \frac{3}{4}
\]

The variance of \( X \):
\[
\sigma^2 = E(X^2) - \mu^2
\]
\[
\sigma^2 = \int_0^1 x^2 \cdot 3x^2 \, dx - \left( \frac{3}{4} \right)^2
\]
\[
= 3 \int_0^1 x^4 \, dx - \frac{9}{16} = \frac{3}{5} - \frac{9}{16} = 0.0375
\]
Problem #9: \( X \sim \text{Normal}(\mu = 31, \sigma^2 = 0.2^2) \)

The standardized Normal \( (0,1) \) is

\[ Z = \frac{X - 31}{0.2} \]

a) \( P(\ x > 30.5\ ) = P(\ Z > \frac{30.5 - 31}{0.2}) = \]
\[ = P(\ Z > -2.5) = 1 - P(\ Z \leq -2.5) = \]
\[ = 1 - \Phi(-2.5) = 1 - (1 - \Phi(2.5)) = \Phi(2.5) = \]
\[ = 0.9938 \]

b) \( P(\ 30.5 < X < 31.5) = P(\frac{30.5 - 31}{0.2} < Z < \frac{31.5 - 31}{0.2}) \)
\[ = P(\ -2.5 < Z < 2.5) = \Phi(2.5) - \Phi(-2.5) = \]
\[ = \Phi(2.5) - (1 - \Phi(2.5)) = 2\Phi(2.5) - 1 = \]
\[ = 2 \times (0.9938) - 1 = 0.9876 \]

c) \( p = P(\ X < 30.4) \) - probability that the tire is under inflated

\[ p = P(\ Z < \frac{30.4 - 31}{0.2}) = P(\ Z < -3) = \Phi(-3) = \]
\[ = 1 - \Phi(3) = 1 - 0.9987 = 0.0013 \]

Let \( Y = \# \) of under inflated tires out of 4

\[ Y \sim \text{Bin}(\ n = 4, \ p = 0.0013) \]
\[ P(\ Y \geq 1) = 1 - P(\ Y = 0) = 1 - \binom{4}{0} 0.0013^0 (1-0.0013)^4 = 0.0052 \]
Let \( Y = X_1 + X_2 \), where

**Problem #10:** \( X_1 \) and \( X_2 \) ~ \( f(x) = 2x \) for \( 0 \leq x \leq 1 \)

(a) \( \mu_x = \int_0^1 x \cdot f(x) \, dx = \int_0^1 x \cdot 2x \, dx = 2 \int_0^1 x^2 \, dx = \frac{2}{3} \cdot x^3 \bigg|_0^1 = \frac{2}{3} \)

\[ \sigma_x^2 = \text{Var}(X) = \mathbb{E}(X - \mu_x)^2 = \mathbb{E}(X^2) - \mu_x^2 = \mathbb{E}(X^2) - \left( \frac{2}{3} \right)^2 \]

But

\[ \mathbb{E}(X^2) = \int_0^1 x^2 \cdot f(x) \, dx = \int_0^1 x^2 \cdot 2x \, dx = 2 \int_0^1 x^3 \, dx = \frac{2}{4} \cdot x^4 \bigg|_0^1 = \frac{2}{4} = \frac{1}{2} \]

Hence,

\[ \sigma_x^2 = \text{Var}(X) = \frac{1}{2} - \left( \frac{2}{3} \right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9 - 8}{18} = \frac{1}{18} \]

(b) Let \( U = \sqrt{18} \cdot (X_1 - \frac{2}{3}) = \sqrt{18} \cdot X_1 - \sqrt{18} \cdot \frac{2}{3} \)

\[ \mathbb{E}(U) = \sqrt{18} \cdot \mathbb{E}(X_1) - \sqrt{18} \cdot \frac{2}{3} = \sqrt{18} \cdot \frac{2}{3} - \sqrt{18} \cdot \frac{2}{3} = 0 \]

\[ \text{Var}(U) = \left( \sqrt{18} \right)^2 \cdot \text{Var}(X_1) = 18 \cdot \frac{1}{18} = 1 \]