1 The Squared Ranks Test for Variances

Data The data consist of two random samples. Let $X_1, X_2, \ldots, X_n$ denote the random sample of size $n$ from population 1 and let $Y_1, Y_2, \ldots, Y_m$, denote the random sample of size $m$ from population 2. Convert each $X_i$ and $Y_j$ to its absolute deviation from the mean using

$$U_i = |X_i - \mu_1|, \quad i = 1, \ldots, n \quad (1)$$

and

$$V_j = |Y_j - \mu_2|, \quad j = 1, \ldots, m \quad (2)$$

where $\mu_1$ and $\mu_2$ are the means for population 1 and 2. If $\mu_1$ and $\mu_2$ are unknown, use $\overline{X}$ for $\mu_1$ and $\overline{Y}$ for $\mu_2$, and the following test is still approximately valid.

Assign the ranks 1 to $n + m$ to the combined sample of $U$s and $V$s in the usual way. If several values of $U$ and/or $V$ are exactly equal to each other (tied), assign to each the average of the ranks that would have been assigned to them had there been no ties. Let $R(U_i)$ and $R(V_j)$ denote the ranks and average ranks thus assigned. Note that ranking the $U_i$s and $V_j$s achieves the same results and is easier than ranking the values of $(X_i - \mu_1)^2$ and $(Y_j - \mu_2)^2$.

Assumptions
1. Both samples are random samples from their respective populations.
2. In addition to independence within each sample there is mutual independence between the two samples.
3. The measurement scale is at least interval.

Test Statistics If there are no values of $U$ tied with values of $V$, the sum of the squares of the ranks assigned to the population 1 can be used as the test statistic.

$$T = \sum_{i=1}^{n}[R(U_i)]^2 \quad (3)$$

If there are ties, subtract the mean from $T$ and divide by the standard deviation to get

$$T_1 = \frac{T - n\overline{R}^2}{\left[\frac{nm}{N(N-1)} \sum_{i=1}^{N} R_i^4 - \frac{nm}{N-1}(\overline{R}^2)^2\right]^{1/2}} \quad (4)$$

where $N = n + m$, $\overline{R}^2$ represents the average of the squared ranks of both samples combined:

$$\overline{R}^2 = \frac{1}{N} \left\{ \sum_{i=1}^{n}[R(U_i)]^2 + \sum_{j=1}^{m}[R(V_j)]^2 \right\} \quad (5)$$

and $\sum R_i^4$ represents the sum of the ranks raised to the fourth power:

$$\sum_{i=1}^{N} R_i^4 = \sum_{i=1}^{n}[R(U_i)]^4 + \sum_{j=1}^{m}[R(V_j)]^4 \quad (6)$$
**Null Distribution** Quantiles of the exact null distribution of $T$ are given in Table A9 for the case of no ties and $n \leq 10$, $m \leq 10$. For sample sizes larger than 10 the following large-sample approximation, based on the standard normal quantile $z_{p}$ given in Table A1, can be used to obtain approximate quantiles $w_{p}$ for $T$.

\[ w_{p} = \frac{n(N + 1)(2N + 1)}{6} + z_{p} \sqrt{\frac{mn(N + 1)(2N + 1)(8N + 11)}{180}} \]  

(7)

Recall that $N = n + m$.

The approximate null distribution $T_{1}$ is the standard normal distribution, Table A1.

**Hypotheses**

A. (Two-Tailed Test)

$H_{0}$: $X$ and $Y$ are identically distributed, except for possibly different means

$H_{1}$: $\text{var}(X) \neq \text{var}(Y)$

Reject $H_{0}$ at the level $\alpha$ if $T$ (or $T_{1}$ in the case of ties) is greater than its $1 - \alpha/2$ quantile or less than its $\alpha/2$ quantile, found from Table A9 or Equation 7 in the case of $T$, or from Table A1 if $T_{1}$ is used. The two-tailed $p$-value is twice the smaller of $P(Z \leq T_{1})$ or $P(Z \geq T_{1})$ obtained directly from Table A1 if $T_{1}$ is used. If $T$ is used the approximate $p$-value can be found using Table A9, to find the smallest two-tailed test that results in rejection of $H_{0}$, or from the normal approximation

\[ p\text{-value} = 2 \cdot (\text{smaller of the one-tailed } p\text{-values}) \]  

(8)

where the lower-tailed $p$-value is approximately

\[ \text{lower-tailed } p\text{-value} = P \left( Z \leq \frac{T - n(N + 1)(2N + 1)/6}{\sqrt{mn(N + 1)(2N + 1)(8N + 11)/180}} \right) \]  

(9)

and the upper-tailed $p$-value is approximately

\[ \text{upper-tailed } p\text{-value} = P \left( Z \geq \frac{T - n(N + 1)(2N + 1)/6}{\sqrt{mn(N + 1)(2N + 1)(8N + 11)/180}} \right) \]  

(10)

B. (Lower-Tailed Test)

$H_{0}$: $X$ and $Y$ are identically distributed, except for possibly different means

$H_{1}$: $\text{var}(X) < \text{var}(Y)$

Reject $H_{0}$ at the level $\alpha$ if $T$ (or $T_{1}$ in the case of ties) is less than its $\alpha$ quantile, found from Table A9 or Equation 7 in the case of $T$, or from Table A1 if $T_{1}$ is used. The $p$-value is the probability of being less than or equal to $T$ (or $T_{1}$) in the null distribution, which is given approximately by Equation 9 for $T$, or by $P(Z \leq T_{1})$, using Table A1.

C. (Upper-Tailed Test)
\( H_0: \) \( X \) and \( Y \) are identically distributed, except for possibly different means
\[ H_1: \quad \text{var}(X) > \text{var}(Y) \]
Reject \( H_0 \) at the level \( \alpha \) if \( T \) (or \( T_1 \) in the case of ties) is greater than its \( 1-\alpha \) quantile, found from Table A9 or Equation 7 in the case of \( T \), or from Table A1 if \( T_1 \) is used. The \( p \)-value is the probability of being greater than or equal to \( T \) (or \( T_1 \)) in the null distribution, which is given approximately by Equation 10 for \( T \), or by \( P(Z \geq T) \), using Table A1.

## 2 A Test for More Than Two Samples

If there are three or more samples, this test is modified easily to test the equality of several variances. From each observation subtract its population mean (or its sample mean when \( \mu_i \) is unknown) and convert the sign of the resulting difference to +, as just described for two samples. Rank the combined absolute differences from smallest to largest, assigning average ranks in case of ties, again as described. Compute the sum of the squares of the ranks of each sample, letting \( S_1, S_2, \ldots, S_k \) denote the sums for each of the \( k \) samples.

\( H_0: \) All \( k \) populations are identical, except for possibly different means
\[ H_1: \quad \text{Some of the population variances are not equal to each other} \]
The test statistic is
\[ T_2 = \frac{1}{D^2} \left[ \sum_{j=1}^{k} \frac{S_j^2}{n_j} - \overline{S}^2 \right] \quad (11) \]
where \( n_j \) = number of observations in sample \( j \), \( N = n_1 + n_2 + \cdots + n_k \), \( S_j \) = the sum of the squared ranks in sample \( j \), \( \overline{S} = \frac{1}{N} \sum_{j=1}^{k} S_j \) = the average of all the squared ranks, \( D^2 = \frac{1}{N-1} \left[ \sum_{i=1}^{N} R_i^4 - N(\overline{S})^2 \right] \), and \( \sum R_i^4 \) represents the sum resulting after raising each rank to the fourth power. If there are no ties, \( D^2 \) and \( \overline{S} \) simplify to
\[ D^2 = \frac{N(N+1)(2N+1)(8N+11)}{180} \quad (12) \]
and
\[ \overline{S} = \frac{(N+1)(2N+1)}{6} \quad (13) \]
The null distribution is approximately the chi-squared distribution with \( k-1 \) degrees of freedom, whose upper quantiles are given in Table A2.
The null hypothesis is rejected if \( T_2 \) exceeds the \( 1-\alpha \) quantile of the chi-squared distribution with \( k-1 \) degrees of freedom, obtained from Table A2.
The \( p \)-value is approximately \( 1 - \text{pchisq}(T_2, df = k - 1) \). If \( H_0 \) is rejected, multiple comparisons may be made as described in the previous section. In this case the variance of the population \( i \) and \( j \) are said to differ if the following inequality is satisfied.
\[ \left| \frac{S_i}{n_i} - \frac{S_j}{n_j} \right| > t_{1-\alpha/2} \left( D^2 \frac{N-1-T_2}{N-k} \right)^{\frac{1}{2}} \left( \frac{1}{n_i} + \frac{1}{n_j} \right)^{\frac{1}{2}} \quad (14) \]
where \( t_{1 - \alpha/2} = qt(1 - \alpha/2, df = N - k) \).

### 3 Theory

Whenever two random variables \( X \) and \( Y \) are identically distributed except for having different means \( \mu_1 \) and \( \mu_2 \), \( X - \mu_1 \) and \( Y - \mu_2 \) not only have zero means, but they are identically distributed also. This means \( U = |X - \mu_1| \) has the same distribution as \( V = |Y - \mu_2| \), and \( U^2 = (X - \mu_1)^2 \) has the same distribution as \( V^2 = (Y - \mu_2)^2 \). So random samples of \( X \)s and \( Y \)s furnish \( U \)s and \( V \)s that are independent and identically distributed. Thus every assignment of ranks of the \( U \)s is equally likely, as in the Mann-Whitney test, and the distribution of any function of the ranks can be found as in the previous section.

### 4 Exercises

5.3.1

```r
> var2.test <- function(X, Y) {
+   n <- length(X)
+   m <- length(Y)
+   N <- n + m
+   U <- abs(X - mean(X))
+   V <- abs(Y - mean(Y))
+   data <- c(U, V)
+   ranks <- rank(data)
+   temp <- sort(data)
+   ties <- F
+   i <- 1
+   while (!ties && i < (N - 1)) {
+     if (temp[i] == temp[i + 1])
+       ties = T
+     i <- i + 1
+   }
+   Tobs <- sum(ranks[1:n]^2)
+   R2bar <- sum(ranks^2)/N
+   T1 <- (Tobs - n * R2bar)/sqrt(n * m/N/(N - 1) * sum(ranks^4) -
+   n * m/(N - 1) * R2bar^2)
+   lp <- pnorm((Tobs - n * (N + 1) * (2 * N + 1)/6)/sqrt(m *
+   n * (N + 1) * (2 * N + 1) * (8 * N + 11)/180))
```
A Test for Equal Variances
Nonparametric Statistics
Fall 2003

+ up <- 1 - lp
+ list(T = Tobs, T1 = T1, ties = ties, lp = lp, up = up)
+
> X <- c(58, 76, 82, 74, 79, 65, 74, 86)
> Y <- c(66, 74, 69, 76, 72, 73, 75, 67, 68)
> var2.test(X, Y)

$T
[1] 1107.5

$T1
[1] 1.390575

$ties
[1] TRUE

$lp
[1] 0.9177113

$up
[1] 0.08228871

It's an upper-tailed test. By Table A9, the critical value is 1159.

5.3.3

> vark.test <- function(data, alpha = 0.05) {
+ k <- length(data)
+ N <- 0
+ n <- rep(1, k)
+ U <- NULL
+ for (j in 1:k) {
+ n[j] <- length(data[[j]])
+ N <- N + n[j]
+ U <- c(U, abs(data[[j]] - mean(data[[j]])))
+ }
+ ranks <- rank(U)
+ pivot <- 0
+ S <- rep(1, k)
+ for (j in 1:k) {
+ }
A Test for Equal Variances

Nonparametric Statistics

Fall 2003

+ $S[j] \leftarrow \sum(\text{ranks}[\text{pivot} + 1:n[j]]^2)$
+ pivot \leftarrow \text{pivot} + n[j]
+ $Sbar \leftarrow \sum(\text{ranks}^2)/N$
+ $D2 \leftarrow (\sum(\text{ranks}^4) - N \times Sbar^2)/(N - 1)$
+ $T2 \leftarrow (\sum(S^2/n) - N \times Sbar^2)/D2$
+ dif \leftarrow \text{matrix}(0, \text{nrow} = k, \text{ncol} = k)$
+ dif.cval \leftarrow \text{matrix}(0, \text{nrow} = k, \text{ncol} = k)$
+ for (i in 1:k) {
+ for (j in 1:k) {
+ dif[i, j] \leftarrow abs(S[i]/n[i] - S[j]/n[j])
+ dif.cval[i, j] \leftarrow qt(1 - \text{alpha}/2, \text{df} = N - k) \times \sqrt(D2 \times (N - 1 - T2)/(N - k)) \times \sqrt(1/n[i] + 1/n[j])
+ }
+ }
+ list(n = n, S = S, N = N, Sbar = Sbar, T2 = T2, D2 = D2,
+ pvalue = 1 - \text{pchisq}(T2, \text{df} = k - 1), difs = dif, difs.cval = dif.cval)
+ }

> X1 <- c(0.7, 1, 2, 1.4, 0.5, 0.8, 1, 1.1, 1.9, 1.2, 1.5)
> X2 <- c(1.7, 2.1, -0.4, 0, 1, 1.1, 0.9, 2.3, 1.3, 0.4, 0.5)
> X3 <- c(0.9, 0.9, 1, 0, 0.1, -0.6, 2.2, -0.3, 0.6, 2.4, 2.5)
> tmp <- vark.test(list(X1, X2, X3))
> tmp$T2

[1] 5.192479

> tmp$pvalue

[1] 0.0745534