1 The Kolmogorov Goodness-of-Fit Test

Data The data consist of a random sample \(X_1, X_2, \ldots, X_n\) of size \(n\) associated with some unknown distribution function, denoted by \(F(x)\).

Assumption
1. The sample is a random sample.

Test Statistic Let \(S(x)\) be the empirical distribution function based on the random sample. The test statistic is defined differently for the three different sets of hypotheses. Let \(F^*(x)\) be a completely specified hypothesized distribution function.

A. (Two-Sided Test)
\[
T = \sup_x |F^*(x) - S(x)|
\]

B. (One-Sided Test)
\[
T^+ = \sup_x [F^*(x) - S(x)]
\]

C. (One-Sided Test)
\[
T^- = \sup_x [S(x) - F^*(x)]
\]

Null Distribution When \(F(x)\) is continuous and the null hypothesis is true the exact distribution function of \(T^+\) and \(T^-\) is given by
\[
G(x) = 1 - x \sum_{j=1}^{\lfloor n(1-x) \rfloor} \binom{n}{j} \left(1 - x - \frac{j}{n}\right)^{n-j} \left(x + \frac{j}{n}\right)^{j-1}
\]

where \(\lfloor n(1-x) \rfloor\) is the greatest integer less than or equal to \(n(1-x)\). This distribution is the same for \(T^+\) and \(T^-\). The asymptotic (as \(n \to \infty\)) distribution function of \(\sqrt{n}T^+\) and \(\sqrt{n}T^-\) is given by
\[
H(x) = \lim_{n \to \infty} G \left( \frac{x}{\sqrt{n}} \right) = 1 - e^{-2x^2}
\]

The approximate distribution function of \(T\) is
\[
P(T \leq x) \approx [G(x)]^2
\]

because \(T\) is less than \(x\) only when both \(T^+\) and \(T^-\) are less than \(x\).

Exact quantiles for \(T\) in the two-sided test, and approximate quantiles for \(T^+\) and \(T^-\) in the one-sided tests, are given in Table A13 for \(n \leq 40\). The asymptotic approximation is used for \(n > 40\). Note that all of these tests are upper-tailed only. The designations “one-sided” and “two-sided” refer to the alternative hypothesis of interest, and the test statistics are refined so that all three tests are upper-tailed.

Table A13 is exact only if \(F(x)\) is continuous; otherwise these quantiles lead to a conservative test.
Hypotheses
A. (Two-Sided Test)
\[ H_0: F(x) = F^*(x) \]
\[ H_1: F(x) \neq F^*(x) \] at some point
Reject \( H_0 \) at the level of significance \( \alpha \) if \( T \) exceeds the \( 1 - \alpha \) quantile as given by Table A13 for the two-tailed test. The approximate \( p \)-value can be found by interpolation in Table A13, or by using twice the one-tailed \( p \)-value given by
\[
\text{one-tailed } p \text{-value} = t \sum_{j=1}^{[n(1-t)j]} \left( \begin{array}{c} n \\ j \end{array} \right) \left( 1 - t - \frac{j}{n} \right)^{n-j} \left( t + \frac{j}{n} \right)^{j-1} \tag{7}
\]
where \( t \) is the observed value of the test statistic.

B. (One-Sided Test)
\[ H_0: F(x) \geq F^*(x) \]
\[ H_1: F(x) < F^*(x) \] at some point
Reject \( H_0 \) at the level of significance \( \alpha \) if \( T^+ \) exceeds the \( (1 - \alpha) \) quantile as given by Table A13 for the one-sided test. The approximate \( p \)-value can be found by interpolation in Table A13. The exact \( p \)-value can be found from Equation 7.

C. (One-Sided Test)
\[ H_0: F(x) \leq F^*(x) \]
\[ H_1: F(x) > F^*(x) \] at some point
Reject \( H_0 \) at the level of significance \( \alpha \) if \( T^- \) exceeds the \( (1 - \alpha) \) quantile as given by Table A13 for the one-sided test. The approximate \( p \)-value can be found by interpolation in Table A13. The exact \( p \)-value can be found from Equation 7.

2 Confidence Band for the Population Distribution Function
\[
L(x) = \max\{0, S(x) - w_{1-\alpha}\}
\]
\[
U(x) = \min\{1, S(x) + w_{1-\alpha}\}
\]

3 Exercises
6.1.1 \( w_{.90} = .509 \)
\[
\text{library(stepfun)}
\]
\[
\text{Fn <- ecdf(c(6.3, 4.2, 4.7, 6, 5.7))}
\]
\[
\text{plot(Fn, verticals = T)}
\]
\[
\text{lines(c(4, 8), c(0, 1))}
\]
6.1.2 $w_{.95} = .454$

> Fn <- ecdf(c(4, 0, 2, 0, 2, 0, 2, 0))
> plot(Fn, verticals = T)
6.1.3 $T = 0.425$, $p$-value $> .20$, Accept.
6.1.4 ???
6.1.5

> Fn <- ecdf(c(1.7, 5.3, 7.6, 8.9, 9, 9.1, 9.3, 9.6, 9.9, 9.9))
> plot(Fn, verticals = T)
> lines(seq(0, 10, by = 0.1), seq(0, 10, by = 0.1)^2/100)
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\[ T = F^*(8.9-) - F_n(8.9-) = 0.4921, \quad p < 0.01, \quad \text{Reject.} \]

6.1.6

\[
\begin{align*}
&\text{> } F_n \leftarrow \text{ecdf(c(4.8, 6.2, 6, 5.9, 6.6, 5.5, 5.8, 5.9, 6.3, 6.6,} \\
&\quad + \quad 6.2, 5)) \\
&\text{> plot(Fn, verticals = T, xlim = qnorm(c(0.1, 0.9), mean = 5.6,} \\
&\quad + \quad \text{sd = 1.2))} \\
&\text{> lines(qnorm(seq(0.1, 0.9, by = 0.02), mean = 5.6, sd = 1.2),} \\
&\quad + \quad \text{seq(0.1, 0.9, by = 0.02))}
\end{align*}
\]
ecdf(c(4.8, 6.2, 6, 5.9, 6.6, 5.5, 5.8, 5.9, 6.3, 6.6, 6.2, 5))

\[ T = F^*(5.8-) - F_n(5.8-) = .3162 \]
\[ .1 < p < .2, \text{ accept.} \]