Chapter 5 Supplemental Text Material

S5-1. Expected Mean Squares in the Two-factor Factorial

Consider the two-factor fixed effects model

$$y_{ij} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1,2,\cdots,a \\ j = 1,2,\cdots,b \\ k = 1,2,\cdots,n \end{cases}$$

given as Equation (5-1) in the textbook. We list the expected mean squares for this model, but do not develop them. It is relatively easy to develop the expected mean squares from direct application of the expectation operator.

Consider finding

$$E(MS_A) = E\left(\frac{SS_A}{a-1}\right) = \frac{1}{a-1}E(SS_A)$$

where SS_A is the sum of squares for the row factor. Since

$$SS_A = \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2 - \frac{y_{...}^2}{abn}$$

$$E(SS_A) = \frac{1}{bn}E\sum_{i=1}^{a} y_{i..}^2 - E\left(\frac{y_{...}^2}{abn}\right)$$

Recall that $\tau_. = 0, \beta_. = 0, (\tau\beta)_{.j} = 0, (\tau\beta)_{i.} = 0,$ and $(\tau\beta)_{..} = 0$, where the "dot" subscript implies summation over that subscript. Now

$$\begin{aligned} y_{i..} &= \sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} = bn\mu + bn\tau_i + n\beta_. + n(\tau\beta)_{i.} + \varepsilon_{i..} \\ &= bn\mu + bn\tau_i + \varepsilon_{i..} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{bn}E\sum_{i=1}^{a} y_{i..}^2 &= \frac{1}{bn}E\sum_{i=1}^{a}\left[(bn\mu)^2 + (bn)^2\tau_i^2 + \varepsilon_{i..}^2 + 2(bn)^2\mu\tau_i + 2bn\mu\varepsilon_{i..} + 2bn\tau_i\varepsilon_{i..}\right] \\ &= \frac{1}{bn}\left[a(bn\mu)^2 + (bn)^2\sum_{i=1}^{a}\tau_i^2 + abn\sigma^2\right] \\ &= abn\mu^2 + bn\sum_{i=1}^{a}\tau_i^2 + a\sigma^2 \end{aligned}$$

Furthermore, we can easily show that

$$y_{...} = abn\mu + \varepsilon_{...}$$

so

$$\begin{aligned}\frac{1}{abn}E(y_{...}^2) &= \frac{1}{abn}E(abn\mu+\varepsilon_{...})^2 \\ &= \frac{1}{abn}E\left[(abn\mu)^2+\varepsilon_{...}^2+2abn\mu\varepsilon_{...})\right] \\ &= \frac{1}{abn}\left[(abn\mu)^2+abn\sigma^2\right] \\ &= abn\mu^2+\sigma^2\end{aligned}$$

Therefore

$$\begin{aligned}E(MS_A) &= E\left(\frac{SS_A}{a-1}\right) \\ &= \frac{1}{a-1}E(SS_A) \\ &\frac{1}{a-1}\left[abn\mu^2+bn\sum_{i=1}^{a}\tau_i^2+a\sigma^2-(abn\mu^2+\sigma^2)\right] \\ &= \frac{1}{a-1}\left[\sigma^2(a-1)+bn\sum_{i=1}^{a}\tau_i^2\right] \\ &= \sigma^2+\frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1}\end{aligned}$$

which is the result given in the textbook. The other expected mean squares are derived similarly.

S5-2. The Definition of Interaction

In Section 5-1 we introduced both the effects model and the means model for the two-factor factorial experiment. If there is no interaction in the two-factor model, then

$$\mu_{ij}=\mu+\tau_i+\beta_j$$

Define the row and column means as

$$\mu_{i.}=\frac{\sum_{j=1}^{b}\mu_{ij}}{b}$$

$$\mu_{.j}=\frac{\sum_{i=1}^{a}\mu_{ij}}{a}$$

Then if there is no interaction,

$$\mu_{ij}=\mu_{i.}+\mu_{.j}-\mu$$

where $\mu = \sum_i \mu_{i.} / a = \sum_j \mu_{.j} / b$. It can also be shown that if there is no interaction, each cell mean can be expressed in terms of three other cell means:

$$\mu_{ij} = \mu_{ij'} + \mu_{i'j} - \mu_{i'j'}$$

This illustrates why a model with no interaction is sometimes called an **additive model**, or why we say the treatment effects are additive.

When there is interaction, the above relationships do not hold. Thus the interaction term $(\tau\beta)_{ij}$ can be defined as

$$(\tau\beta)_{ij} = \mu_{ij} - (\mu + \tau_i + \beta_j)$$

or equivalently,

$$\begin{aligned}(\tau\beta)_{ij} &= \mu_{ij} - (\mu_{ij'} + \mu_{i'j} - \mu_{i'j'}) \\ &= \mu_{ij} - \mu_{ij'} - \mu_{i'j} + \mu_{i'j'}\end{aligned}$$

Therefore, we can determine whether there is interaction by determining whether all the cell means can be expressed as $\mu_{ij} = \mu + \tau_i + \beta_j$.

Sometimes interactions are a result of the **scale** on which the response has been measured. Suppose, for example, that factor effects act in a multiplicative fashion,

$$\mu_{ij} = \mu\tau_i\beta_j$$

If we were to assume that the factors act in an additive manner, we would discover very quickly that there is interaction present. This interaction can be removed by applying a log transformation, since

$$\log \mu_{ij} = \log \mu + \log \tau_i + \log \beta_j$$

This suggests that the original measurement scale for the response was not the best one to use if we want results that are easy to interpret (that is, no interaction). The log scale for the response variable would be more appropriate.

Finally, we observe that it is very possible for two factors to interact but for the main effects for one (or even both) factor is small, near zero. To illustrate, consider the two-factor factorial with interaction in Figure 5-1 of the textbook. We have already noted that the interaction is large, *AB* = -29. However, the main effect of factor *A* is *A* = 1. Thus, the main effect of *A* is so small as to be negligible. Now this situation does not occur all that frequently, and typically we find that interaction effects are not larger than the main effects. However, large two-factor interactions can **mask** one or both of the main effects. A prudent experimenter needs to be alert to this possibility.

S5-3. Estimable Functions in the Two-factor Factorial Model

The least squares normal equations for the two-factor factorial model are given in Equation (5-14) in the textbook as:

$$abn\hat{\mu}+bn\sum_{i=1}^{a}\hat{\tau}_i+an\sum_{j=1}^{b}\beta_j+\sum_{i=1}^{a}\sum_{j=1}^{b}(\hat{\tau\beta})_{ij}=y_{...}$$

$$bn\hat{\mu}+bn\hat{\tau}_i+n\sum_{j=1}^{b}\beta_j+n\sum_{j=1}^{b}(\hat{\tau\beta})_{ij}=y_{i..}, i=1,2,\cdots,a$$

$$an\hat{\mu}+n\sum_{i=1}^{a}\hat{\tau}_i+an\hat{\beta}_j+n\sum_{i=1}^{a}(\hat{\tau\beta})_{ij}=y_{.j.}, j=1,2,\cdots,b$$

$$n\hat{\mu}+n\hat{\tau}_i+n\hat{\beta}_j+n(\hat{\tau\beta})_{ij}=y_{ij.}, \begin{cases} i=1,2,\cdots,a \\ j=1,2,\cdots,b \end{cases}$$

Recall that in general an estimable function must be a linear combination of the left-hand side of the normal equations. Consider a contrast comparing the effects of row treatments *i* and i'. The contrast is

$$\tau_i-\tau_{i'}+(\overline{\tau\beta})_{i.}-(\overline{\tau\beta})_{i'.}$$

Since this is just the difference between two normal equations, it is an estimable function. Notice that the difference in any two levels of the row factor also includes the difference in average interaction effects in those two rows. Similarly, we can show that the difference in any pair of column treatments also includes the difference in average interaction effects in those two columns. An estimable function involving interactions is

$$(\tau\beta)_{ij}-(\overline{\tau\beta})_{i.}-(\overline{\tau\beta})_{.j}+(\overline{\tau\beta})_{..}$$

It turns out that the only hypotheses that can be tested in an effects model must involve estimable functions. Therefore, when we test the hypothesis of no interaction, we are really testing the null hypothesis

$$H_0\colon(\tau\beta)_{ij}-(\overline{\tau\beta})_{i.}-(\overline{\tau\beta})_{.j}+(\overline{\tau\beta})_{..}=0 \text{ for all } i, j$$

When we test hypotheses on main effects *A* and *B* we are really testing the null hypotheses

$$H_0\colon\tau_1+(\overline{\tau\beta})_{1.}=\tau_2+(\overline{\tau\beta})_{2.}=\cdots=\tau_a+(\overline{\tau\beta})_{a.}$$

and

$$H_0\colon\beta_1+(\overline{\tau\beta})_{.1}=\beta_2+(\overline{\tau\beta})_{.2}=\cdots=\beta_b+(\overline{\tau\beta})_{.b}$$

That is, we are not really testing a hypothesis that involves only the equality of the treatment effects, but instead a hypothesis that compares treatment effects *plus* the average interaction effects in those rows or columns. Clearly, these hypotheses may not be of much interest, or much practical value, when interaction is large. This is why in the textbook (Section 5-1) that when interaction is large, main effects may not be of much practical value. Also, when interaction is large, the statistical tests on main effects may not really tell us much about the individual treatment effects. Some statisticians do not even conduct the main effect tests when the no-interaction null hypothesis is rejected.

It can be shown [see Myers and Milton (1991)] that the original effects model

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$$

can be re-expressed as

$$\begin{aligned} y_{ijk} = {} & [\mu + \bar{\tau} + \bar{\beta} + (\overline{\tau\beta})] + [\tau_i - \bar{\tau} + (\overline{\tau\beta})_{i.} - (\overline{\tau\beta})] + \\ & [\beta_j - \bar{\beta} + (\overline{\tau\beta})_{.j} - (\overline{\tau\beta})] + [(\tau\beta)_{ij} - (\overline{\tau\beta})_{i.} - (\overline{\tau\beta})_{.j} + (\overline{\tau\beta})] + \varepsilon_{ijk} \end{aligned}$$

or

$$y_{ijk} = \mu^* + \tau_i^* + \beta_j^* + (\tau\beta)_{ij}^* + \varepsilon_{ijk}$$

It can be shown that each of the new parameters $\mu^*, \tau_i^*, \beta_j^*,$ and $(\tau\beta)_{ij}^*$ is estimable. Therefore, it is reasonable to expect that the hypotheses of interest can be expressed simply in terms of these redefined parameters. It particular, it can be shown that there is no interaction if and only if $(\tau\beta)_{ij}^* = 0$. Now in the text, we presented the null hypothesis of no interaction as $H_0{:}(\tau\beta)_{ij} = 0$ for all *i* and *j*. This is not incorrect so long as it is understood that it is the model in terms of the redefined (or “starred”) parameters that we are using. However, it is important to understand that in general interaction is *not* a parameter that refers only to the (*ij*)th cell, but it contains information from that cell, the *i*th row, the *j*th column, and the overall average response.

One final point is that as a consequence of defining the new “starred” parameters, we have included certain restrictions on them. In particular, we have

$$\tau_.^* = 0, \beta_.^* = 0,\ (\tau\beta)_{i.}^* = 0,\ (\tau\beta)_{.j}^* = 0 \text{ and } (\tau\beta)_{..}^* = 0$$

These are the “usual constraints” imposed on the normal equations. Furthermore, the tests on main effects become

$$H_0{:}\tau_1^* = \tau_2^* = \cdots = \tau_a^* = 0$$

and

$$H_0{:}\beta_1^* = \beta_2^* = \cdots = \beta_b^* = 0$$

This is the way that these hypotheses are stated in the textbook, but of course, without the “stars”.

S5-4. Regression Model Formulation of the Two-factor Factorial

We noted in Chapter 3 that there was a close relationship between ANOVA and regression, and in the Supplemental Text Material for Chapter 3 we showed how the single-factor ANOVA model could be formulated as a regression model. We now show how the two-factor model can be formulated as a regression model and a standard multiple regression computer program employed to perform the usual ANOVA.

We will use the battery life experiment of Example 5-1 to illustrate the procedure. Recall that there are three material types of interest (factor *A*) and three temperatures (factor *B*), and the response variable of interest is battery life. The regression model formulation of an ANOVA model uses **indicator variables**. We will define the indicator variables for the design factors material types and temperature as follows:

Material type	X_1	X_2
1	0	0
2	1	0
3	0	1

Temperature	X_3	X_4
15	0	0
70	1	0
125	0	1

The regression model is

$$\begin{aligned} y_{ijk} = {} & \beta_0 + \beta_1 x_{ijk1} + \beta_2 x_{ijk2} + \beta_3 x_{ijk3} + \beta_4 x_{ijk4} \\ & + \beta_5 x_{ijk1} x_{ijk3} + \beta_6 x_{ijk1} x_{ijk4} + \beta_7 x_{ijk2} x_{ijk3} + \beta_8 x_{ijk2} x_{ijk4} + \varepsilon_{ijk} \end{aligned} \tag{1}$$

where i, j =1,2,3 and the number of replicates k = 1,2,3,4. In this model, the terms $\beta_1 x_{ijk1} + \beta_2 x_{ijk2}$ represent the main effect of factor *A* (material type), and the terms $\beta_3 x_{ijk3} + \beta_4 x_{ijk4}$ represent the main effect of temperature. Each of these two groups of terms contains two regression coefficients, giving two degrees of freedom. The terms $\beta_5 x_{ijk1} x_{ijk3} + \beta_6 x_{ijk1} x_{ijk4} + \beta_7 x_{ijk2} x_{ijk3} + \beta_8 x_{ijk2} x_{ijk4}$ in Equation (1) represent the *AB* interaction with four degrees of freedom. Notice that there are four regression coefficients in this term.

Table 1 shows the data from this experiment, originally presented in Table 5-1 of the text. In Table 1, we have shown the indicator variables for each of the 36 trials of this experiment. The notation in this table is $X_i = x_i$, i=1,2,3,4 for the main effects in the above regression model and $X_5 = x_1 x_3$, $X_6 = x_1 x_4$, $X_7 = x_2 x_3$, and $X_8 = x_2 x_4$, for the interaction terms in the model.

Table 1. Data from Example 5-1 in Regression Model Form

Y	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
130	0	0	0	0	0	0	0	0
34	0	0	1	0	0	0	0	0
20	0	0	0	1	0	0	0	0
150	1	0	0	0	0	0	0	0
136	1	0	1	0	1	0	0	0
25	1	0	0	1	0	1	0	0
138	0	1	0	0	0	0	0	0
174	0	1	1	0	0	0	1	0
96	0	1	0	1	0	0	0	1
155	0	0	0	0	0	0	0	0
40	0	0	1	0	0	0	0	0
70	0	0	0	1	0	0	0	0
188	1	0	0	0	0	0	0	0
122	1	0	1	0	1	0	0	0
70	1	0	0	1	0	1	0	0
110	0	1	0	0	0	0	0	0
120	0	1	1	0	0	0	1	0
104	0	1	0	1	0	0	0	1
74	0	0	0	0	0	0	0	0
80	0	0	1	0	0	0	0	0
82	0	0	0	1	0	0	0	0
159	1	0	0	0	0	0	0	0
106	1	0	1	0	1	0	0	0
58	1	0	0	1	0	1	0	0
168	0	1	0	0	0	0	0	0
150	0	1	1	0	0	0	1	0
82	0	1	0	1	0	0	0	1
180	0	0	0	0	0	0	0	0
75	0	0	1	0	0	0	0	0
58	0	0	0	1	0	0	0	0
126	1	0	0	0	0	0	0	0
115	1	0	1	0	1	0	0	0
45	1	0	0	1	0	1	0	0
160	0	1	0	0	0	0	0	0
139	0	1	1	0	0	0	1	0
60	0	1	0	1	0	0	0	1

This table was used as input to the Minitab regression procedure, which produced the following results for fitting Equation (1):

Regression Analysis

```
The regression equation is
y = 135 + 21.0 x1 + 9.2 x2 - 77.5 x3 - 77.2 x4 + 41.5 x5 - 29.0 x6
 +79.2 x7 + 18.7 x8
```

```
minitab Output (Continued)

Predictor        Coef       StDev          T        P
Constant       134.75       12.99      10.37    0.000
x1              21.00       18.37       1.14    0.263
x2               9.25       18.37       0.50    0.619
x3             -77.50       18.37      -4.22    0.000
x4             -77.25       18.37      -4.20    0.000
x5              41.50       25.98       1.60    0.122
x6             -29.00       25.98      -1.12    0.274
x7              79.25       25.98       3.05    0.005
x8              18.75       25.98       0.72    0.477

S = 25.98       R-Sq = 76.5%     R-Sq(adj) = 69.6%

Analysis of Variance

Source             DF          SS          MS         F        P
Regression          8     59416.2      7427.0     11.00    0.000
Residual Error     27     18230.7       675.2
Total              35     77647.0

Source       DF      Seq SS
x1            1       141.7
x2            1     10542.0
x3            1        76.1
x4            1     39042.7
x5            1       788.7
x6            1      1963.5
x7            1      6510.0
x8            1       351.6
```

First examine the Analysis of Variance information in the above display. Notice that the regression sum of squares with 8 degrees of freedom is equal to the sum of the sums of squares for the main effects material types and temperature and the interaction sum of squares from Table 5-5 in the textbook. Furthermore, the number of degrees of freedom for regression (8) is the sum of the degrees of freedom for main effects and interaction (2 +2 + 4) from Table 5-5. The *F*-test in the above ANOVA display can be thought of as testing the null hypothesis that *all* of the model coefficients are zero; that is, there are no significant main effects or interaction effects, versus the alternative that there is at least one nonzero model parameter. Clearly this hypothesis is rejected. Some of the treatments produce significant effects.

Now consider the "sequential sums of squares" at the bottom of the above display. Recall that X_1 and X_2 represent the main effect of material types. The sequential sums of squares are computed based on an "effects added in order" approach, where the "in order" refers to the order in which the variables are listed in the model. Now

$$SS_{MaterialTypes} = SS(X_1) + SS(X_2|X_1) = 141.7 + 10542.0 = 10683.7$$

which is the sum of squares for material types in table 5-5. The notation $SS(X_2|X_1)$ indicates that this is a "sequential" sum of squares; that is, it is the sum of squares for variable X_2 given that variable X_1 is already in the regression model.

Similarly,

$$SS_{Temperature} = SS(X_3|X_1, X_2) + SS(X_4|X_1, X_2, X_3) = 76.1 + 39042.7 = 39118.8$$

which closely agrees with the sum of squares for temperature from Table 5-5. Finally, note that the interaction sum of squares from Table 5-5 is

$$\begin{aligned} SS_{Interaction} &= SS(X_5|X_1, X_1, X_3, X_4) + SS(X_6|X_1, X_1, X_3, X_4, X_5) \\ &+ SS(X_7|X_1, X_2, X_3, X_4, X_5, X_6) + SS(X_8|X_1, X_2, X_3, X_4, X_5, X_6, X_7) \\ &= 788.7 + 1963.5 + 6510.0 + 351.6 = 9613.8 \end{aligned}$$

When the design is **balanced**, that is, we have an equal number of observations in each cell, we can show that this model regression approach using the sequential sums of squares produces results that are exactly identical to the "usual" ANOVA. Furthermore, because of the balanced nature of the design, the order of the variables *A* and *B* does not matter.

The "effects added in order" partitioning of the overall model sum of squares is sometimes called a **Type 1 analysis**. This terminology is prevalent in the SAS statistics package, but other authors and software systems also use it. An alternative partitioning is to consider each effect as if it were added **last** to a model that contains all the others. This "effects added last" approach is usually called a **Type 3 analysis**.

There is another way to use the regression model formulation of the two-factor factorial to generate the standard *F*-tests for main effects and interaction. Consider fitting the model in Equation (1), and let the regression sum of squares in the Minitab output above for this model be the model sum of squares for the **full model**. Thus,

$$SS_{Model}(\text{FM}) = 59416.2$$

with 8 degrees of freedom. Suppose we want to test the hypothesis that there is no interaction. In terms of model (1), the no-interaction hypothesis is

$$\begin{aligned} &H_0{:}\beta_5 = \beta_6 = \beta_7 = \beta_8 = 0 \\ &H_0{:}\ \text{at least one } \beta_j \neq 0,\ j = 5,6,7,8 \end{aligned} \tag{2}$$

When the null hypothesis is true, a **reduced model** is

$$y_{ijk} = \beta_0 + \beta_1 x_{ijk1} + \beta_2 x_{ijk2} + \beta_3 x_{ijk3} + \beta_4 x_{ijk4} + \varepsilon_{ijk} \tag{3}$$

Fitting Equation (2) using Minitab produces the following:

```
The regression equation is
y = 122 + 25.2 x1 + 41.9 x2 - 37.3 x3 - 80.7 x4
```

Predictor	Coef	StDev	T	P
Constant	122.47	11.17	10.97	0.000
x1	25.17	12.24	2.06	0.048
x2	41.92	12.24	3.43	0.002
x3	-37.25	12.24	-3.04	0.005
x4	-80.67	12.24	-6.59	0.000

```
S = 29.97       R-Sq = 64.1%     R-Sq(adj) = 59.5%

Analysis of Variance
```

Source	DF	SS	MS	F	P
Regression	4	49802	12451	13.86	0.000
Residual Error	31	27845	898		
Total	35	77647			

The model sum of squares for this reduced model is

$$SS_{Model}(\text{RM}) = 49802.0$$

with 4 degrees of freedom. The test of the no-interaction hypotheses (2) is conducted using the "extra" sum of squares

$$\begin{aligned} SS_{Model}(\text{Interaction}) &= SS_{Model}(\text{FM}) - SS_{Model}(\text{RM}) \\ &= 59{,}416.2 - 49{,}812.0 \\ &= 9{,}604.2 \end{aligned}$$

with 8 – 4 = 4 degrees of freedom. This quantity is, apart from round-off errors in the way the results are reported in Minitab, the interaction sum of squares for the original analysis of variance in Table 5-5 of the text. This is a measure of fitting interaction after fitting the main effects.

Now consider testing for no main effect of material type. In terms of equation (1), the hypotheses are

$$\begin{aligned} &H_0{:}\beta_1 = \beta_2 = 0 \\ &H_0{:}\ \text{at least one } \beta_j \neq 0,\ j = 1,2 \end{aligned} \tag{4}$$

Because we are using a balanced design, it turns out that to test this hypothesis all we have to do is to fit the model

$$y_{ijk} = \beta_0 + \beta_1 x_{ijk1} + \beta_2 x_{ijk2} + \varepsilon_{ijk} \tag{5}$$

Fitting this model in Minitab produces

Regression Analysis

```
The regression equation is
y = 83.2 + 25.2 x1 + 41.9 x2
```

Predictor	Coef	StDev	T	P
Constant	83.17	13.00	6.40	0.000
x1	25.17	18.39	1.37	0.180
x2	41.92	18.39	2.28	0.029

```
Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2       10684        5342      2.63    0.087
Residual Error    33       66963        2029
Total             35       77647
```

Notice that the regression sum of squares for this model [Equation (5)] is essentially identical to the sum of squares for material types in table 5-5 of the text. Similarly, testing that there is no temperature effect is equivalent to testing

$$\begin{aligned} &H_0{:}\beta_3 = \beta_4 = 0 \\ &H_0{:}\ \text{at least one } \beta_j \neq 0,\ j = 3{,}4 \end{aligned} \tag{6}$$

To test the hypotheses in (6), all we have to do is fit the model

$$y_{ijk} = \beta_0 + \beta_3 x_{ijk3} + \beta_4 x_{ijk4} + \varepsilon_{ijk} \tag{7}$$

The Minitab regression output is

Regression Analysis

```
The regression equation is
y = 145 - 37.3 x3 - 80.7 x4

Predictor        Coef       StDev           T        P
Constant      144.833       9.864       14.68    0.000
x3             -37.25       13.95       -2.67    0.012
x4             -80.67       13.95       -5.78    0.000

S = 34.17       R-Sq = 50.4%     R-Sq(adj) = 47.4%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         2       39119       19559     16.75    0.000
Residual Error    33       38528        1168
Total             35       77647
```

Notice that the regression sum of squares for this model, Equation (7), is essentially equal to the temperature main effect sum of squares from Table 5-5.

S5-5. Model Hierarchy

In Example 5-4 we used the data from the battery life experiment (Example 5-1) to demonstrate fitting response curves when one of the factors in a two-factor factorial experiment was quantitative and the other was qualitative. In this case the factors are temperature (*A*) and material type (*B*). Using the Design-Expert software package, we fit a model that the main effect of material type, the linear and quadratic effects of temperature, the material type by linear effect of temperature interaction, and the material type by quadratic effect of temperature interaction. Refer to Table 5-15 in the textbook. From examining this table, we observed that the quadratic effect of temperature and the material type by linear effect of temperature interaction were not significant; that is, they had fairly large *P*-values. We left these non-significant terms in the model to preserve hierarchy.

The hierarchy principal states that if a model contains a higher-order term, then it should also contain all the terms of lower-order that comprise it. So, if a second-order term, such as an interaction, is in the model then all main effects involved in that interaction as well as all lower-order interactions involving those factors should also be included in the model.

There are times that hierarchy makes sense. Generally, if the model is going to be used for explanatory purposes then a hierarchical model is quite logical. On the other hand, there may be situations where the non-hierarchical model is much more logical. To illustrate, consider another analysis of Example 5-4 in Table 2, which was obtained from Design-Expert. We have selected a non-hierarchical model in which the quadratic effect of temperature was not included (it was in all likelihood the weakest effect), but both two-degree-of-freedom components of the temperature-material type interaction are in the model. Notice from Table 2 that the residual mean square is smaller for the non-hierarchical model (653.81 versus 675.21 from Table 5-15). This is important, because the residual mean square can be thought of as the variance of the unexplained residual variability, not accounted for by the model. That is, the non-hierarchical model is actually a *better fit* to the experimental data.

Notice also that the standard errors of the model parameters are smaller for the non-hierarchical model. This is an indication that he parameters are estimated with better precision by leaving out the nonsignificant terms, even though it results in a model that does not obey the hierarchy principal. Furthermore, note that the 95 percent confidence intervals for the model parameters in the hierarchical model are always longer than their corresponding confidence intervals in the non-hierarchical model. The non-hierarchical model, in this example, does indeed provide better estimates of the factor effects that obtained from the hierarchical model

Table 2. Design-Expert Output for Non-hierarchical Model, Example 5-4.

ANOVA for Response Surface Reduced Cubic Model
Analysis of variance table [Partial sum of squares]

Source	**Sum of Squares**	**DF**	**Mean Square**	**F Value**	**Prob > F**
Model	59340.17	7	8477.17	12.97	< 0.0001
A	*10683.72*	*2*	*5341.86*	*8.17*	*0.0016*
B	*39042.67*	*1*	*39042.67*	*59.72*	*< 0.0001*
AB	*2315.08*	*2*	*1157.54*	*1.77*	*0.1888*
AB^2	*7298.69*	*2*	*3649.35*	*5.58*	*0.0091*
Residual	18306.81	28	653.81		
Lack of Fit	*76.06*	*1*	*76.06*	*0.11*	*0.7398*
Pure Error	*18230.75*	*27*	*675.21*		
Cor Total	77646.97	35			

Std. Dev.	25.57	R-Squared	0.7642
Mean	105.53	Adj R-Squared	0.7053
C.V.	24.23	Pred R-Squared	0.6042
PRESS	30729.09	Adeq Precision	8.815

Term	**Coefficient Estimate**	**DF**	**Standard Error**	**95% CI Low**	**95% CI High**
Intercept	105.53	1	4.26	96.80	114.26
A[1]	-50.33	1	10.44	-71.72	-28.95
A[2]	12.17	1	10.44	-9.22	33.55
B-Temp	-40.33	1	5.22	-51.02	-29.64
A[1]B	1.71	1	7.38	-13.41	16.83
A[2]B	-12.79	1	7.38	-27.91	2.33
A[1]B^2	41.96	1	12.78	15.77	68.15
A[2]B^2	-14.04	1	12.78	-40.23	12.15

Supplemental Reference

Myers, R. H. and Milton, J. S. (1991), *A First Course in the Theory of the Linear Model*, PWS-Kent, Boston, MA.