

## Chapter 7. Supplemental Text Material

### S7-1. The Error Term in a Blocked Design

Just as in any randomized complete block design, when we run a replicated factorial experiment in blocks we are assuming that there is no interaction between treatments and blocks. In the RCBD with a single design factor (Chapter 4) the error term is actually the interaction between treatments and blocks. This is also the case in a factorial design. To illustrate, consider the ANOVA in Table 7-2 of the textbook. The design is a  $2^2$  factorial run in three complete blocks. Each block corresponds to a replicate of the experiment. There are six degrees of freedom for error. Two of those degrees of freedom are the interaction between blocks and factor  $A$ , two degrees of freedom are the interaction between blocks and factor  $B$ , and two degrees of freedom are the interaction between blocks and the  $AB$  interaction. In order for the error term here to truly represent random error, we must assume that blocks and the design factors do not interact.

### S7-2. The Prediction Equation for a Blocked Design

Consider the prediction equation for the  $2^4$  factorial in two blocks with  $ABCD$  confounded from in Example 7-2. Since blocking does not impact the effect estimates from this experiment, the equation would be exactly the same as the one obtained from the unblocked design, Example 6-2. This prediction equation is

$$\hat{y} = 70.06 + 10.8125x_1 + 4.9375x_3 + 7.3125x_4 - 9.0625x_1x_3 + 8.3125x_1x_4$$

This equation would be used to predict *future* observations where we had no knowledge of the block effect. However, in the experiment just completed we know that there is a strong block effect, in fact the block effect was computed as

$$block\ effect = \bar{y}_{block1} - \bar{y}_{block2} = -18.625$$

This means that the difference in average response between the two blocks is  $-18.625$ . We should compensate for this in the prediction equation if we want to obtain the correct fitted values for block 1 and block 2. Defining a separate block effect for each block does this, where  $block_1\ effect = -9.3125$  and  $block_2\ effect = 9.3125$ . These block effects would be added to the intercept in the prediction equation for each block. Thus the prediction equations are

$$\begin{aligned}\hat{y}_{block_1} &= 70.06 + block_1\ effect + 10.8125x_1 + 4.9375x_3 + 7.3125x_4 - 9.0625x_1x_3 + 8.3125x_1x_4 \\ &= 70.06 + (-9.3125) + 10.8125x_1 + 4.9375x_3 + 7.3125x_4 - 9.0625x_1x_3 + 8.3125x_1x_4 \\ &= 60.7475 + 10.8125x_1 + 4.9375x_3 + 7.3125x_4 - 9.0625x_1x_3 + 8.3125x_1x_4\end{aligned}$$

and

$$\begin{aligned}\hat{y}_{block_2} &= 70.06 + block_2\ effect + 10.8125x_1 + 4.9375x_3 + 7.3125x_4 - 9.0625x_1x_3 + 8.3125x_1x_4 \\ &= 70.06 + (9.3125) + 10.8125x_1 + 4.9375x_3 + 7.3125x_4 - 9.0625x_1x_3 + 8.3125x_1x_4 \\ &= 79.3725 + 10.8125x_1 + 4.9375x_3 + 7.3125x_4 - 9.0625x_1x_3 + 8.3125x_1x_4\end{aligned}$$

### S7-3. Run Order is Important

Blocking is really all about experimental run order. Specifically, we run an experiment in blocks to provide protection against the effects of a known and controllable nuisance factor(s). However, in many experimental situations, it is a good idea to conduct the experiment in blocks, even though there is no obvious nuisance factor present. This is particularly important when it takes several time periods (days, shifts, weeks, etc.) to run the experiment.

To illustrate, suppose that we are conducting a single replicate of a  $2^4$  factorial design. The experiment is shown in run order is shown in Table 2. Now suppose that misfortune strikes the experimenter, and after the first eight trials have been performed it becomes impossible to complete the experiment. Is there any useful experimental design that can be formed from the first eight runs?

**Table 2.** A  $2^4$  Factorial Experiment

Std Order	Run Order	Block	Factor A	Factor B	Factor C	Factor D
2	1	Block 1	1	-1	-1	-1
12	2	Block 1	1	1	-1	1
10	3	Block 1	1	-1	-1	1
15	4	Block 1	-1	1	1	1
14	5	Block 1	1	-1	1	1
4	6	Block 1	1	1	-1	-1
7	7	Block 1	-1	1	1	-1
3	8	Block 1	-1	1	-1	-1
5	9	Block 1	-1	-1	1	-1
8	10	Block 1	1	1	1	-1
11	11	Block 1	-1	1	-1	1
16	12	Block 1	1	1	1	1
1	13	Block 1	-1	-1	-1	-1
9	14	Block 1	-1	-1	-1	1
6	15	Block 1	1	-1	1	-1
13	16	Block 1	-1	-1	1	1

It turns out that in this case, the answer to that question is “no”. Now some analysis can of course be performed, but it would basically consist of fitting a regression model to the response data from the first 8 trials. Suppose that we fit a regression model containing an intercept term and the four main effects. When things have gone wrong it is usually a good idea to focus on simple objectives, making use of the data that are available. It turns out that in that model we would actually be obtaining estimates of

$$[\text{Intercept}] = \text{Intercept} - AB + CD - ABCD$$

$$[A] = A + AB - BC - ABC + ACD - BCD$$

$$\begin{aligned}
[B] &= B + AB - BC - ABC \\
[C] &= C - ABC + ACD - BCD \\
[D] &= D - ABD - ACD + BCD
\end{aligned}$$

Now suppose we feel comfortable in ignoring the three-factor and four-factor interaction effects. However, even with these assumptions, our intercept term is “clouded” or “confused” with two of the two-factor interactions, and the main effects of factors *A* and *B* are “confused” with the other two-factor interactions. In the next chapter, we will refer to the phenomena being observed here as **aliasing** of effects (its proper name). The supplemental notes for Chapter 8 present a general method for deriving the aliases for the factor effects. The Design-Expert software package can also be used to generate the aliases by employing the Design Evaluation feature. Notice that in our example, not completing the experiment as originally planned has really disturbed the interpretation of the results.

Suppose that instead of completely randomizing all 16 runs, the experimenter had set this  $2^4$  design up in two blocks of 8 runs each, selecting in the usual way the *ABCD* interaction to be confounded with blocks. Now if only the first 8 runs can be performed, then it turns out that the estimates of the intercept and main factor effects from these 8 runs are

$$\begin{aligned}
[\text{Intercept}] &= \text{Intercept} \\
[A] &= A + BCD \\
[B] &= B + ACD \\
[C] &= C + ABD \\
[D] &= D + ABC
\end{aligned}$$

If we assume that the three-factor interactions are negligible, then we have reliable estimates of all four main effects from the first 8 runs. The reason for this is that each block of this design forms a **one-half fraction** of the  $2^4$  factorial, and this fraction allows estimation of the four main effects free of any two-factor interaction aliasing. This specific design (the one-half fraction of the  $2^4$ ) will be discussed in considerable detail in Chapter 8.

This illustration points out the importance of thinking carefully about run order, even when the experimenter is not obviously concerned about nuisance variables and blocking. Remember:

If something can go wrong when conducting an experiment, it probably will.  
A prudent experimenter designs his or her experiment with this in mind.

Generally, if a  $2^k$  factorial design is constructed in two blocks, and one of the blocks is lost, ruined, or never run, the  $2^k / 2 = 2^{k-1}$  runs that remain will always form a **one-half fraction** of the original design. It is almost always possible to learn *something* useful from such an experiment.

To take this general idea a bit further, suppose that we had originally set up the 16-run  $2^4$  factorial experiment in four blocks of four runs each. The design that we would obtain using the standard methods from this chapter in the text gives the experiment in Table 3. Now suppose that for some reason we can only run the first 8 trials of this experiment. It is easy to verify that the first 8 trials in Table 3 **do not** form one of the usual 8-run blocks produced by confounding the *ABCD* interaction with blocks. Therefore, the first 8 runs in Table 3 are not a “standard” one-half fraction of the  $2^4$ .

A logical question is “what can we do with these 8 runs?” Suppose, as before, that the experimenter elects to concentrate on estimating the main effects. If we use only the first eight runs from Table 3 and concentrate on estimating only the four main effects, it turns out what we *really* are estimating is

$$\begin{aligned} [\text{Intercept}] &= \text{Intercept} - \text{ACD} \\ [A] &= A - \text{CD} \\ [B] &= B - \text{ABCD} \\ [C] &= C - \text{AD} \\ [D] &= D - \text{AC} \end{aligned}$$

Once again, even assuming that all interactions beyond order two are negligible, our main effect estimates are aliased with two-factor interactions.

**Table 3.** A  $2^4$  Factorial Experiment in Four Blocks

Std Order	Run Order	Block	Factor A	Factor B	Factor C	Factor D
10	1	Block 1	1	-1	-1	1
15	2	Block 1	-1	1	1	1
3	3	Block 1	-1	1	-1	-1
6	4	Block 1	1	-1	1	-1
12	5	Block 2	1	1	-1	1
8	6	Block 2	1	1	1	-1
13	7	Block 2	-1	-1	1	1
1	8	Block 2	-1	-1	-1	-1
11	9	Block 3	-1	1	-1	1
2	10	Block 3	1	-1	-1	-1
7	11	Block 3	-1	1	1	-1
14	12	Block 3	1	-1	1	1
16	13	Block 4	1	1	1	1
5	14	Block 4	-1	-1	1	-1
9	15	Block 4	-1	-1	-1	1
4	16	Block 4	1	1	-1	-1

If we were able to obtain 12 of the original 16 runs (that is, the first *three* blocks of Table 3), then we can estimate

$$[\text{Intercept}] = \text{Intercept} - 0.333 * AB - 0.333 * ACD - 0.333 * BCD$$

$$[A] = A - ABCD$$

$$[B] = B - ABCD$$

$$[C] = C - ABC$$

$$[D] = D - ABD$$

$$[AC] = AC - ABD$$

$$[AD] = AD - ABC$$

$$[BC] = BC - ABD$$

$$[BD] = BD - ABC$$

$$[CD] = CD - ABCD$$

If we can ignore three- and four-factor interactions, then we can obtain good estimates of all four main effects and five of the six two-factor interactions. Once again, setting up and running the experiment in blocks has proven to be a good idea, even though no nuisance factor was anticipated. Finally, we note that it is possible to assemble three of the four blocks from Table 3 to obtain a 12-run experiment that is slightly better than the one illustrated above. This would actually be called a  $3/4^{\text{th}}$  fraction of the  $2^4$ , an *irregular* fractional factorial. These designs are mentioned briefly in the Chapter 8 exercises.