Chapter 13. Supplemental Text Material

S13-1. Expected Mean Squares for the Random Model

We consider the two-factor random effects balanced ANOVA model

$$y_{ij} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1,2,\cdots,a \\ j = 1,2,\cdots,b \\ k = 1,2,\cdots,n \end{cases}$$

given as Equation (13-15) in the textbook. We list the expected mean squares for this model in Equation (13-17), but do not formally develop them. It is relatively easy to develop the expected mean squares from direct application of the expectation operator.

For example, consider finding

$$E(MS_A) = E\left(\frac{SS_A}{a-1}\right) = \frac{1}{a-1}E(SS_A)$$

where SS_A is the sum of squares for the row factor. Recall that the model components τ_i, β_j and $(\tau\beta)_{ij}$ are normally and independently distributed with means zero and variances $\sigma_\tau^2, \sigma_\beta^2,$ and $\sigma_{\tau\beta}^2$ respectively. The sum of squares and its expectation are defined as

$$SS_A = \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2 - \frac{y_{...}^2}{abn}$$

$$E(SS_A) = \frac{1}{bn}E\sum_{i=1}^{a} y_{i..}^2 - E\left(\frac{y_{...}^2}{abn}\right)$$

Now

$$y_{i..} = \sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} = bn\mu + bn\tau_i + n\beta_. + n(\tau\beta)_{i.} + \varepsilon_{i..}$$

and

$$\begin{aligned}\frac{1}{bn}E\sum_{i=1}^{a} y_{i..}^2 &= \frac{1}{bn}E\sum_{i=1}^{a}\left[(bn\mu)^2 + (bn)^2\tau_i^2 + \varepsilon_{i..}^2 + 2(bn)^2\mu\tau_i + 2bn\mu\varepsilon_{i..} + 2bn\tau_i\varepsilon_{i..}\right] \\ &= \frac{1}{bn}\left[a(bn\mu)^2 + a(bn)^2\sigma_\tau^2 + ab(n)^2\sigma_\beta^2 + abn^2\sigma_{\tau\beta}^2 + abn\sigma^2\right] \\ &= abn\mu^2 + abn\sigma_\tau^2 + an\sigma_\beta^2 + an\sigma_{\tau\beta}^2 + a\sigma^2\end{aligned}$$

Furthermore, we can show that

$$y_{...} = abn\mu + bn\tau_. + an\beta_. + n(\tau\beta)_{..} + \varepsilon_{...}$$

so the second term in the expected value of SS_A becomes

$$\frac{1}{abn}E(y_{...}^2) = \frac{1}{abn}\left[(abn\mu)^2 + a(bn)^2\sigma_\tau^2 + b(an)^2\sigma_\beta^2 + abn^2\sigma_{\tau\beta}^2 + abn\sigma^2\right]$$
$$= abn\mu^2 + bn\sigma_\tau^2 + an\sigma_\beta^2 + n\sigma_{\tau\beta}^2 + \sigma^2$$

We can now collect the components of the expected value of the sum of squares for factor *A* and find the expected mean square as follows:

$$E(MS_A) = E\left(\frac{SS_A}{a-1}\right)$$
$$\frac{1}{a-1}\left[\frac{1}{bn}E\sum_{i=1}^{a} y_{i..}^2 - E\left(\frac{y_{...}^2}{abn}\right)\right]$$
$$= \frac{1}{a-1}\left[\sigma^2(a-1) + n(a-1)\sigma_{\tau\beta}^2 + bn\sigma_\tau^2\right]$$
$$= \sigma^2 + n\sigma_{\tau\beta}^2 + bn\sigma_\tau^2$$

This agrees with the first result in Equation (15-17).

S13-2. Expected Mean Squares for the Mixed Model

As noted in Section 13-3 of the textbook, there are several version of the mixed model, and the expected mean squares depend on which model assumptions are used. In this section, we assume that the **restricted model** is of interest. The next section considers the unrestricted model.

Recall that in the restricted model there are assumptions made regarding the fixed factor, *A*; namely,

$$\tau_{.} = 0, (\tau\beta)_{.j} = 0 \text{ and } V[(\tau\beta)_{ij}] = \left[\frac{a}{a-1}\right]\sigma_{\tau\beta}^2$$

We will find the expected mean square for the random factor, *B*. Now

$$E(MS_B) = E\left(\frac{SS_B}{b-1}\right)$$
$$= \frac{1}{b-1}E(SS_B)$$

and

$$E(SS_B) = \frac{1}{an}E\sum_{j=1}^{b} y_{.j.}^2 - \frac{1}{abn}E(y_{...}^2)$$

Using the restrictions on the model parameters, we can easily show that

$$y_{.j.} = an\mu + an\beta_j + \varepsilon_{.j.}$$

and

$$\frac{1}{an}E\sum_{j=1}^{b}y_{.j.}^2 = \left[b(an\mu)^2 + b(an)^2\sigma_\beta^2 + abn\sigma^2\right]$$
$$= abn\mu^2 + abn\sigma_\beta^2 + b\sigma^2$$

Since

$$y_{...} = abn\mu + an\beta_{.} + \varepsilon_{...}$$

we can easily show that

$$\frac{1}{abn}E(y_{...}^2) = \frac{1}{abn}\left[(abn\mu)^2 + b(an)^2\sigma_\beta^2 + abn\sigma^2\right]$$
$$= abn\mu^2 + an\sigma_\beta^2 + \sigma^2$$

Therefore the expected value of the mean square for the random effect is

$$\begin{aligned} E\left(MS_B\right) &= \frac{1}{b-1}E\left(SS_B\right) \\ &= \frac{1}{b-1}\left(abn\mu^2 + abn\sigma_\beta^2 + b\sigma^2 - abn\mu^2 - an\sigma_\beta^2 - \sigma^2\right) \\ &= \frac{1}{b-1}\left[\sigma^2(b-1) + an(b-1)\sigma_\beta^2\right] \\ &= \sigma^2 + an\sigma_\beta^2 \end{aligned}$$

The other expected mean squares can be derived similarly.

S13-3. Restricted versus Unrestricted Mixed Models

We now consider the **unrestricted model**

$$y_{ij} = \mu + \alpha_i + \gamma_j + (\alpha\gamma)_{ij} + \varepsilon_{ijk}\begin{cases} i = 1,2,\cdots,a \\ j = 1,2,\cdots,b \\ k = 1,2,\cdots,n \end{cases}$$

for which the assumptions are

$$\alpha_{.} = 0 \text{ and } V[(\alpha\gamma)_{ij}] = \sigma_{\alpha\gamma}^2$$

and all random effects are uncorrelated random variables. Notice that there is no assumption concerning the interaction effects summed over the levels of the fixed factor as is customarily made for the restricted model. Recall that the restricted model is actually a more general model than the unrestricted model, but some modern computer programs give the user a choice of models (and some computer programs only use the unrestricted model), so there is increasing interest in both versions of the mixed model.

We will derive the expected value of the mean square for the random factor, *B*, in Equation (13-26), as it is different from the corresponding expected mean square in the restricted model case. As we will see, the assumptions regarding the interaction effects are instrumental in the difference in the two expected mean squares.

The expected mean square for the random factor, *B*, is defined as

$$E(MS_B) = E\left(\frac{SS_B}{b-1}\right)$$
$$= \frac{1}{b-1}E(SS_B)$$

and, as in the cases above

$$E(SS_B) = \frac{1}{an}E\sum_{j=1}^{b} y_{.j.}^2 - \frac{1}{abn}E(y_{...}^2)$$

First consider

$$y_{.j.} = \sum_{i=1}^{a}\sum_{k=1}^{n} y_{ijk} = an\mu + n\alpha_{.} + an\gamma_{.} + n(\alpha\gamma)_{.j} + \varepsilon_{.j.}$$
$$= an\mu + an\gamma_{.} + n(\alpha\gamma)_{.j} + \varepsilon_{.j.}$$

because $\alpha_{.} = 0$. Notice, however, that the interaction term in this expression is *not* zero as it would be in the case of the restricted model. Now the expected value of the first part of the expression for $E(SS_B)$ is

$$\frac{1}{an}E\sum_{j=1}^{b} y_{.j.}^2 = \frac{1}{an}\left[b(an\mu)^2 + b(an)^2\sigma_{\gamma}^2 + abn^2\sigma_{\alpha\gamma}^2 + abn\sigma^2\right]$$
$$= abn\mu^2 + abn\sigma_{\gamma}^2 + bn\sigma_{\alpha\gamma}^2 + b\sigma^2$$

Now we can show that

$$y_{...} = abn\mu + bn\alpha_{.} + an\gamma_{.} + n(\alpha\gamma)_{..} + \varepsilon_{...}$$
$$= abn\mu + an\gamma_{.} + n(\alpha\gamma)_{..} + \varepsilon_{...}$$

Therefore

$$\frac{1}{abn}E(y_{...}^2) = \frac{1}{abn}\left[(abn\mu)^2 + b(an)^2\sigma_{\gamma}^2 + abn^2\sigma_{\alpha\gamma}^2 + abn\sigma^2\right]$$
$$= abn\mu^2 + an\sigma_{\gamma}^2 + n\sigma_{\alpha\gamma}^2 + \sigma^2$$

We may now assemble the components of the expected value of the sum of squares for factor *B* and find the expected value of MS_B as follows:

$$
\begin{aligned}
E\left(MS_B\right) &= \frac{1}{b-1}E\left(SS_B\right) \\
&= \frac{1}{b-1}\left[\frac{1}{an}E\sum_{j=1}^{b} y_{.j.}^2 - \frac{1}{abn}E(y_{...}^2)\right] \\
&= \frac{1}{b-1}\left[abn\mu^2 + abn\sigma_\gamma^2 + bn\sigma_{\alpha\gamma}^2 + b\sigma^2 - \left(abn\mu^2 + an\sigma_\gamma^2 + n\sigma_{\alpha\gamma}^2 + \sigma^2\right)\right] \\
&= \frac{1}{b-1}[\sigma^2(b-1) + n(b-1)\sigma_{\alpha\gamma}^2 + an(b-1)\sigma_\gamma^2] \\
&= \sigma^2 + n\sigma_{\alpha\gamma}^2 + an\sigma_\gamma^2
\end{aligned}
$$

This last expression is in agreement with the result given in Equation (13-26).

Deriving expected mean squares by the direct application of the expectation operator (the "brute force" method) is tedious, and the rules given in the text are a great labor-saving convenience. There are other rules and techniques for deriving expected mean squares, including algorithms that will work for unbalanced designs. See Milliken and Johnson (1984) for a good discussion of some of these procedures.

S13-4. Random and Mixed Models with Unequal Sample Sizes

Generally, ANOVA models become more complicated to analyze when the designs is unbalanced; that is, when some of the cells contain different numbers of observations. In Chapter 15, we briefly discuss this problem in the two-factor fixed-effects design. The unbalanced case of random and mixed models is not discussed there, but we offer some very brief advice in this section.

An unbalanced random or mixed model will not usually have exact *F*-tests as they did in the balanced case. Furthermore, the Satterthwaite approximate or synthetic *F*-test does not apply to unbalanced designs. The simplest approach to the analysis is based on the method of maximum likelihood. This approach to variance component estimation was discussed in Section 13-7.3, and the SAS procedure employed there can be used for unbalanced designs. The disadvantage of this approach is that all the inference on variance components is based on the maximum likelihood large sample theory, which is only an approximation because designed experiments typically do not have a large number of runs. The book by Searle (1987) is a good reference on this general topic.

S13-5. Some Background Concerning the Modified Large Sample Method

In Section 12-7.2 we discuss the modified large same method for determining a confidence interval on variance components that can be expressed as a linear combination of mean squares. The large sample theory essentially states that

$$
Z = \frac{\hat{\sigma}_0^2 - \sigma_0^2}{\sqrt{V(\hat{\sigma}_0^2)}}
$$

has a normal distribution with mean zero and variance unity as $\min(f_1, f_2, \cdots, f_Q)$ approaches infinity, where

$$V(\hat{\sigma}_0^2) = 2\sum_{i=1}^{Q} c_i^2 \theta_i^2 / f_i \,,$$

θ_i is the linear combination of variance components estimated by the *i*th mean square, and f_i is the number of degrees of freedom for MS_i. Consequently, the 100(1-α) percent large-sample two-sided confidence interval for σ_0^2 is

$$\hat{\sigma}_0^2 - z_{\alpha/2}\sqrt{V(\hat{\sigma}_0^2)} \le \sigma_0^2 \le \hat{\sigma}_0^2 + z_{\alpha/2}\sqrt{V\hat{\sigma}_0^2)}$$

Operationally, we would replace θ_i by MS_i in actually computing the confidence interval. This is the same basis used for construction of the confidence intervals by SAS PROC MIXED that we presented in section 13-7.3 (refer to the discussion of tables 13-17 and 13-18 in the textbook).

These large-sample intervals work well when the number of degrees of freedom are large, but when the f_i are small they may be unreliable. However, the performance may be improved by applying suitable modifications to the procedure. Welch (1956) suggested a modification to the large-sample method that resulted in some improvement, but Graybill and Wang (1980) proposed a technique that makes the confidence interval exact for certain special cases. It turns out that it is also a very good approximate procedure for the cases where it is not an exact confidence interval. Their result is given in the textbook as Equation (13-42).

S13-6. A Confidence Interval on a Ratio of Variance Components using the Modified Large Sample Method

As observed in the textbook, the modified large sample method can be used to determine confidence intervals on ratios of variance components. Such confidence intervals are often of interest in practice. For example, consider the measurement systems capability study described in Example 12-2 in the textbook. In this experiment, the total variability from the gauge is the sum of three variance components $\sigma_\beta^2 + \sigma_{\tau\beta}^2 + \sigma^2$, and the variability of the product used in the experiment is σ_τ^2. One way to describe the capability of the measurement system is to present the variability of the gauge as a percent of the product variability. Therefore, an experimenter would be interested in the ratio of variance components

$$\frac{\sigma_\beta^2 + \sigma_{\tau\beta}^2 + \sigma^2}{\sigma_\tau^2}$$

Suppose that σ_1^2 / σ_2^2 is a ratio of variance components of interest and that we can estimate the variances in the ratio by the ratio of two linear combinations of mean squares, say

$$\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{\sum_{i=1}^{P} c_i MS_i}{\sum_{j=P+1}^{Q} c_j MS_j}$$

Then a 100(1-α) percent lower confidence in interval on σ_1^2 / σ_2^2 is given by

$$L = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2}\left[\frac{2 + k_4/(k_1k_2) - \sqrt{V_L}}{2(1 - k_5/k_2^2)}\right]$$

where

$$V_L = (2 + k_4/(k_1k_2))^2 - 4(1 - k_5/k_2^2)(1 - k_3/k_1^2)$$

$$k_1 = \sum_{i=1}^{P} c_i MS_i, \quad k_2 = \sum_{j=P+1}^{Q} c_j MS_j$$

$$k_3 = \sum_{i=1}^{P} G_i^2 c_i^2 MS_i^2 + \sum_{i=1}^{P-1}\sum_{t>i}^{P} G_{it}^* c_i c_t MS_i MS_t$$

$$k_4 = \sum_{i=1}^{P}\sum_{j=P+1}^{Q} G_{ij} c_1 c_j MS_i MS_j$$

$$k_5 = \sum_{j=P+1}^{Q} H_j^2 c_j^2 MS_j^2$$

and G_i, H_j, G_{ig}, and G_{it}^* are as previously defined. For more details, see the book by Burdick and Graybill (1992).

Supplemental References

Graybill, F. A. and C. M. Wang (1980. “Confidence Intervals on Nonnegative Linear Combinations of Variances”. *Journal of the American Statistical Association*, Vol. 75, pp. 869-873.

Welch, B. L. (1956). “On Linear Combinations of Several Variances”. *Journal of the American Statistical Association*, Vol. 51, pp. 132-148.